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A-POSTERIORI ERROR ESTIMATES AND ADAPTIVE TECHNIQUES FOR THE FI--ETC(U)

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ABSTRACT

The theory of the computable a-posteriori error estimate for a finite element method is developed. Among other things, it is shown that the error estimate is very reliable and the ratio (called effectivity index) between the estimator and the true error approaches one. Numerical examples computed by program FEARS (Finite Element Adaptive Research Solver) of the University of Maryland, illustrate the effectivity and reliability of the estimators.

A

1. INTRODUCTION

Recently an increasing interest in the finite element computations is being focused on the reliability of the results and the quality of the used meshes and elements.

During recent years at the University of Maryland, the studies were undertaken which focused toward the development of a finite element system having the following features.

a) The solver supplies the user with a reliable and accurate information about achieved accuracy in the desired norm.

b) The solver constructs adaptively meshes which are leading to the highest possible accuracy (through an adaptive refinement).

c) The solver uses the most simple input.

d) The solver combines the advances in the mathematics and computer science including parallel computations.

The solver FEARS (Finite Element Adaptive Research Solver), its mathematical version FM developed for Univac Series 1100 implements some of the points mentioned above*. The detailed description of FEARS and the experience with it will be published elsewhere. For some information about FEARS and its applications, we refer to [2], [3], [4], [5].

*For analysis of the parallelity we refer to [1].

One of the main aspects of FEARS is the theory of the a-posteriori estimates. Some aspects of the a-posteriori error estimates and optimality of the meshes were investigated in [6], [7], [8].

The error norm $||e||$ is approximated by the computable estimator ε computed through error indicators $\eta_i(\Delta)$ associated to every element Δ and computable locally by knowledge of the finite element solution at the particular element Δ and its direct neighbors. The effectivity index $\theta = \frac{\varepsilon}{||e||}$ expresses the quality of the estimator and θ should be close to one when the error is sufficiently small (e.g. 5%). It is desirable that the estimator ε has the following two properties:

$$(1.1) \qquad 0 < C_L \leq \theta \leq C_U < \infty$$

with C_L and C_U independent of the solution and the meshes under very general conditions.

$$(1.2) \qquad \theta \to 1 \quad \text{as} \quad ||e|| \to 0$$

provided that some additional assumptions about smoothness are made. The present paper develops the theory of the estimator which satisfies (1.1) and (1.2), and is implemented in FEARS. The energy error norm is assumed and model elasticity problem is considered.

Section 2 consists of some preliminary notions.

Sections 3 and 4 elaborate on the type of meshes which are adaptively constructed.

Section 5 deals with the approximation properties of the elements on the

admissible meshes.

Section 6 formulates the model problem (elasticity problem).

Section 7 develops the estimator and proves (1.1).

Section 8 proves that the estimator is asymptotically correct, i.e. $\theta \to 1$.

Section 9 deals with two computational examples and discusses the effectivity of the approach.

The adaptive construction of the meshes is based on the equilibration of the error indicators. This principle was theoretically analyzed in [7] for one dimensional problems and its theoretical investigation in the context of FEARS will appear elsewhere.

2. BASIC NOTATION

Throughout this article we denote by R^2 the two dimensional real Euclidian space with $x \equiv (x_1,x_2) \in \mathbb{R}^2$, $||x|| = \max(|x_1|,|x_2|), ||x||_E = [x_1^2+x_2^2]^{1/2}$. Let $Q \subset R^2$ be a bounded set and ∂Q its boundary. We define

$$\operatorname{diam} Q = \sup_{x,y\in Q} ||x-y|| \quad ,$$

$$\operatorname{dist}(x,Q) = \inf_{y\in Q} ||x-y||$$

and for $Q_i \in R^2$, $i = 1,2$

$$\operatorname{dist}(Q_1,Q_2) = \inf_{\substack{x\in Q_1 \\ x\in Q_2}} ||x-y|| \quad .$$

An index E will denote that the norm $||\cdot||_E$ is used instead of $||\cdot||$. E.g., $\operatorname{dist}_E(x,Q) = \inf_{y\in Q} ||x-y||_E$.

For a $\rho > 0$, Q^ρ is the ρ-neighborhood of Q

$$Q^\rho = \{x\in\mathbb{R}^2 | \operatorname{dist}(x,Q) < \rho \} .$$

The closure of Q in $\mathbb{R}_2$ is denoted by $\overline{Q}$, int Q means as usual the interior of Q .

By Z^2 we denote the set of all two dimensional integers $k \equiv (k_1,k_2)$, k_i, $i = 1,2$ integral.

Suppose $\theta > 0$ is a positive real number, then we will write for any $k \in Z^2$

$$Q_\theta^k = \{x \in \mathbb{R}^2 \mid k_i\theta \leq x_i \leq (k_i+1)\theta \ , \quad i = 1,2\} \quad .$$

Assume that $Z_o \subset Z^2$ is a finite set. Then we denote

$$\Omega_{Z_o,\theta} = \mathrm{int}[\bigcup_{k \in Z_o} Q_\theta^k] \quad .$$

We shall assume that Z_o is such that $\Omega_{Z_o,\theta}$ is a Lipschitz domain. For brevity, whenever it cannot lead to misunderstanding we shall write Ω instead of $\Omega_{Z_o,\theta}$. When we talk of a square S in $\mathbb{R}^2$, we shall always suppose that it is closed and that its sides are parallel to the coordinate axes, i.e. S is of the form $[a,a+d] \times [b,b+d]$ for $a,b,d \in \mathbb{R}$, $d > 0$.

As usual, let $L_2(\Omega) = H^o(\Omega)$ be the space of all square integrable functions on Ω with the inner product

$$(u,v)_{L_2(\Omega)} = \int_\Omega uv\,dx, \quad dx = dx_1 dx_2$$

and the corresponding norm $||\cdot||_{L_2(\Omega)}$. By $H^k(\Omega)$, $k \geq 0$ integral we denote the usual Sobolev space with the norm

$$||u||^2_{H^k(\Omega)} = \sum_{0 \leq |\alpha| \leq k} ||D^\alpha u||_{L_2(\Omega)} \quad ,$$

where $\alpha = (\alpha_1,\alpha_2)$, $\alpha_i \geq 0$, $|\alpha| = \alpha_1 + \alpha_2$

and

$$D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \quad .$$

Obviously we have $H^o(\Omega) = L_2(\Omega)$. We will also use the notation

$$|u|^2_{H^k(\Omega)} = \sum_{|\alpha|=k} ||D^\alpha u||^2_{L_2(\Omega)} \quad .$$

We define the support of a function $u \in L_2(\Omega)$ in the usual (distributive) way and denote it by supp u . Let $\overset{\circ}{H}{}^k(\Omega) \subset H^k(\Omega)$ be the completion of the set of all functions having compact support in Ω .

We will also deal with functions defined on one dimensional manifolds, more precisely on the boundary $\partial\Omega$ or a part Γ of it. The notation $L_2(\Gamma) = H^o(\Gamma)$ has then the obvious meaning.

Let $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$, where each Γ_i is a closed side of some $Q_\theta^{k^{(i)}} \subset \Omega$, $k^{(i)} \in Z_o$ with $\Gamma_i \subset \partial\Omega$, $(i = 1,\ldots,m)$; then we shall write

$$H^{1,\Gamma}(\Omega) = \{u \in H^1(\Omega)\ u = 0 \text{ on } \Gamma\} \quad .$$

Obviously $\overset{\circ}{H}{}^1(\Omega) = H^{1,\Gamma}(\Omega)$ when $\Gamma = \partial\Omega$ and $H^1(\Omega) = H^{1,\Gamma}(\Omega)$ when $\Gamma = \emptyset$.

Finally by $C^o(\Omega)$ we denote the space of all continuous functions on $\overline{\Omega}$ and let

$$||u||_{C^o(\Omega)} = \sup_{x\in\Omega} |u(x)| \quad .$$

We will deal later with extensions of functions in $H^1(\Omega)$, $\overset{\circ}{H}{}^1(\Omega)$ and $H^{1,\Gamma}(\Omega)$ from Ω into a neighborhood of Ω .

<u>Theorem 2.1.</u> <u>There exists an operator T mapping $H^{1,\Gamma}(\Omega)$ into $H^1(\Omega^\rho)$, (where Ω^ρ is a ρ-neighborhood of Ω) such that</u>

i) for a $2 < \beta < \infty$, $0 < \alpha_o < \rho$ and any square $S \subset \Omega$ and any $0 < \alpha < \alpha_o$ we have

$$||Tu||_{H^1(S^\alpha)} \leq C||u||_{H^1(S^{\beta\alpha} \cap \Omega)}$$

with C independent of α, S, u.

ii) If $x \in \Omega^\rho$, $x \notin \Omega$, $\Gamma \neq \partial\Omega$ and $\mathrm{dist}_E(x,\Gamma) \leq \frac{1}{10}\,\mathrm{dist}_E(x,\partial\Omega-\Gamma)$ then

$$Tu = 0 \quad \text{on} \quad S^\gamma(x)-\Omega, \quad \gamma = \frac{3}{2}\,\mathrm{dist}_E(x,\Gamma)$$

with $S^\gamma(x)$ being the square with the center in x and lengthside γ.

iii) If $\Gamma = \partial\Omega$ i.e., $H^{1,\Gamma}(\Omega) = \mathring{H}^1(\Omega)$, then $Tu = 0$ on $\Omega^\rho - \Omega$.

Proof. If is enough to prove the theorem in the neighborhood of the boundary (i.e., endpoints) of Γ. In the neighborhood of all other points $x \in \partial\Omega$ we use the classical extension theorem when $x \notin \Gamma$ and we extend by zero when $x \in \Gamma$ and apply the standard argument with partition of unity.

The endpoint of Γ can be located in a vertex of $\partial\Omega$ with the internal angle $\frac{3}{2}\pi$ or $\frac{1}{2}\pi$ or it can be on straight part of the boundary.

We will deal only with the case of the vertex in the coordinates origin and the internal angle on $\frac{3}{2}\pi$. The other cases are analogous.

Let (see Fig. 2.1)

$$\Omega_*^\rho = \{x \in \Omega^\rho \mid x \notin \Omega,\ \mathrm{dist}_E(x,\partial\Omega-\Gamma) \leq \frac{1}{2}\,\mathrm{dist}_E(x,\Gamma)\}$$

$$\Omega^{\rho}_{**} = \{x \in \Omega^{\rho} \mid x \notin \Omega^{\rho}_{*},\ \mathrm{dist}_E(x,\Gamma) < \frac{1}{2}\,\mathrm{dist}_E(x,\partial\Omega-\Gamma)\}\ ,$$

$$\Omega^{\rho}_{***} = \Omega^{\rho} - \Omega^{\rho}_{**} - \Omega^{\rho}_{*} - \Gamma\ .$$

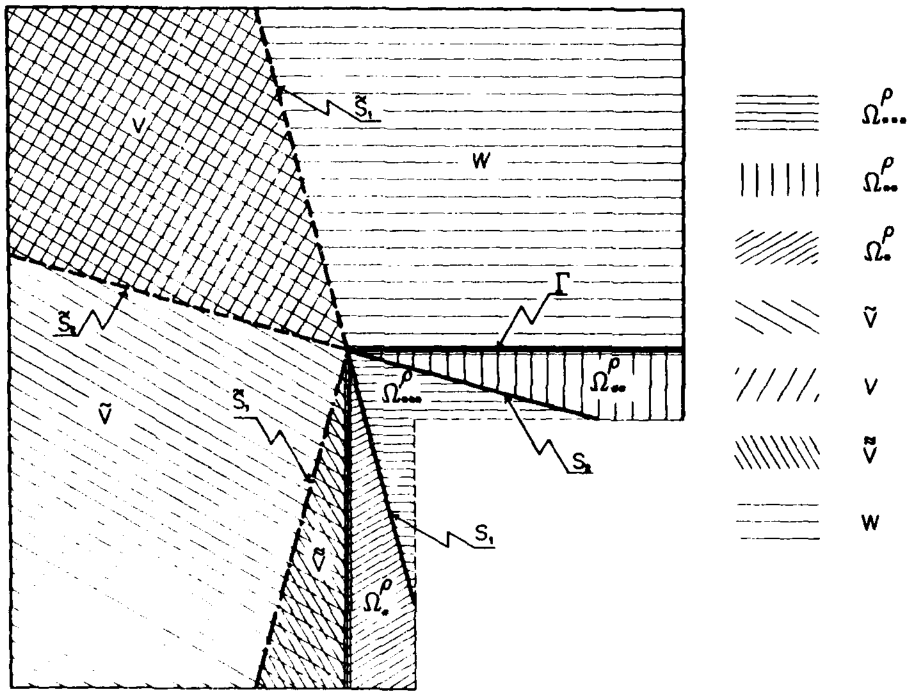


Figure 2.1

The scheme of notation in the proof of the Theorem 2.1.

Obviously Ω^{Ω}_{***} consists of a sector with the lines S_1 and S_2 as its boundary. Let V be a symmetric (with respect to the origin) sector with the boundary $\tilde{S}_1$ and $\tilde{S}_2$ and let $\tilde{\tilde{S}}_1$ be the line symmetric to S_1 (with respect to $\partial\Omega - \Gamma$) . Finally let $\tilde{V}\subset\Omega$ (respectively $\tilde{\tilde{V}}\subset\Omega$) be the sector bounded by $\partial\Omega - \Gamma$ and $\tilde{S}^1$ (respectively $\tilde{\tilde{S}}_1$) and W be the sector bounded by $\tilde{S}^1$ and Γ .

Assume now that $u\in H^{1,\Gamma}(\Omega)$. By an afine transformation, we construct $w\in H^1(\tilde{\tilde{V}})$ such that $w = u$ on $\partial\Omega - \Gamma$, $w(x) = u(\hat{x})$ for $x\in\tilde{\tilde{S}}_1$, $\hat{x}\in\tilde{S}_1$ and $||x||_E = ||\hat{x}||_E$. Obviously

$$||w||_{H^1(\tilde{\tilde{V}})} \le C||u||_{H^1(\tilde{V})} .$$

The extention Tu on Ω^{Γ}_{*} be now the reflection of w (with respect to $\partial\Omega - \Gamma$) . It is readily seen that

$$||Tu||_{H^1(\Omega^{\Omega}_{*})} \le ||w||_{H^1(\tilde{\tilde{V}})} \le C||u||_{H^1(\tilde{V})}$$

and for

$$x\in S_1,\quad \hat{x}\in\tilde{S}_1,\quad ||x||_E = ||\hat{x}||_E$$

we have

$$(Tu)(x) = u(\hat{x}) .$$

We extend u on Ω^{Γ}_{**} by zero.

By an afine transformation of W on V we can easily continue a function v such that $v = u$ on $\tilde{S}_1$ and $v = 0$ on $\tilde{S}_2$ and

$$||v||_{H^1(V)} \leq C||u||_{H^1(W)} \quad .$$

Now let Tu on Ω^{ρ}_{***} be the symmetric image of v. It is easy to see that our construction has all properties of the extension formulated in the theorem when ρ is chosen sufficiently large.

3. THE MESH AND ITS BASIC PROPERTIES

We will introduce now a class of partitions of $\Omega_{Z_o,0}$.

We define a mesh $\mathcal{D}(\Omega) = \{\Delta^i\}$ as a finite collection of closed squares $\Delta^i \subset \bar{\Omega}$ of various sizes with sides parallel to the coordinate axes, and which are generated by the following recursive rules.

i) The squares $\{Q_0^k\}$, $k \in Z_o$, create a mesh.

ii) If $\{\Delta^i\}$, $i = 1,\ldots,m$ is a mesh, then a new mesh is obtained if any Δ^i is subdivided into four congruent squares of half the side length of Δ^i .

Any $\Delta^i \in \mathcal{D}$ will be called an element and its sides the edges. The vertices of the elements will be called the nodes. A node P will be called a regular node if either $P \in \partial\Omega$ or P is a vertex of four different elements. Otherwise P is an irregular mode. By $P(\mathcal{D})$ we denote the set of all nodes of $\mathcal{D}$ and by $R(\mathcal{D}) \subset P(\mathcal{D})$ the set of all regular nodes. Finally let $h(\mathcal{D}) = \max_{\Delta \in \mathcal{D}} \operatorname{diam} \Delta$.

Figure 3.1 shows an example of a mesh. The irregular nodes are marked by a cross.

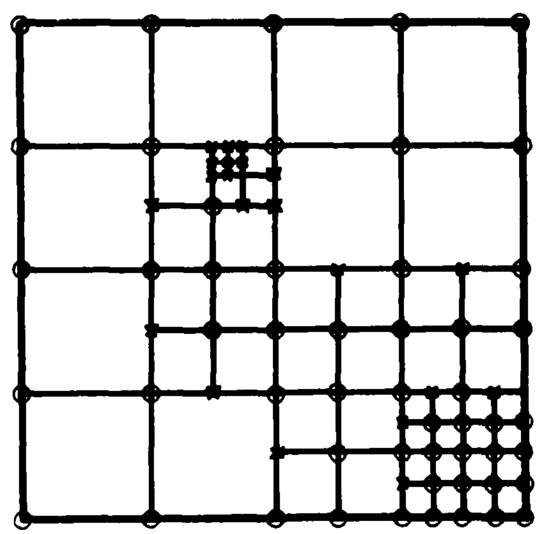


Figure 3.1. An example of a mesh.

Let $\mathcal{D} = \{\Delta^i\}$, $i = 1,\ldots,m$. Then obviously $\bigcup_{i=1}^{m} \Delta^i = \bar{\Omega}$. We shall denote by $M(\mathcal{D})$, the subspace of all continuous functions on $\bar{\Omega}$ which are individually bilinear on each Δ^i, $i = 1,\ldots,m$. It is clear that $M(\mathcal{D}) \subset H^1(\Omega)$.

We will always assume that at least four different $\Delta \in \mathcal{D}$ lie in every Q_Ω^k.

Now we will analyze further the basic properties of the meshes introduced above.

LEMMA 3.1. Let $\Delta', \Delta'' \in \mathcal{D}$.

1) Assume that $\Delta' \cap \Delta'' \neq \emptyset$ and $\operatorname{diam} \Delta' \leq \operatorname{diam} \Delta''$. Then one and only one of the following statements holds.

i) $\Delta' = \Delta''$

ii) $\Delta' \cap \Delta''$ is just one point being a common vertex of Δ' and Δ''.

iii) $\Delta' \cap \Delta''$ is an edge of Δ' and is contained in an edge, Γ say, of Δ''. If x', x'' are any two endpoints of $(\Delta' \cap \Delta'')$ and Γ respectively, then $|x'-x''|$ is an integral multiple of $\operatorname{diam} \Delta'$.

2) If $P \in \mathcal{P}(\mathcal{D})$, $P \notin R(\mathcal{D})$ (i.e., P is an irregular mode) and P is a vertex of $\Delta' \in \mathcal{D}$, then there exists $\Delta'' \in \mathcal{D}$ such that

i) $P \in \Delta''$

ii) $\operatorname{diam} \Delta'' > \operatorname{diam} \Delta'$

iii) $\Delta'' \cap \Delta'$ is an edge of Δ'.

3) $\operatorname{diam}\Delta''/\operatorname{diam}\Delta' = 2^s$ with s an integer.

The lemma can be easily proven by induction.

LEMMA 3.2. Suppose that $\Delta \in \mathcal{D}$. Then at least one vertex of Δ is a

regular node and if an edge of Δ is contained in $\partial\Omega_\theta^k$ for some $k \in Z_o$, then at least one vertex of Δ which is not on $\partial\Omega_\theta^k$ is a regular node.

Proof. Observe that any regular node always remains regular when our recurrent construction is implemented. The lemma follows now easily since at each step the midpoint of the subdivided element becomes a regular node and it is a vertex of all four new elements created at that step.

As seen in Figure 3.1 there could be an element $\Delta \in \mathcal{D}$, such that only one of its four vertices is a regular node.

LEMMA 3.3. Let $\mathcal{D}$ be a mesh and $P \in \mathcal{R}(\mathcal{D})$. Then there exists $v_p \in \mathcal{M}(\mathcal{D})$ such that $v_p(P) = 1$ and $v_p(Q) = 0$ for any $Q \in \mathcal{R}(\mathcal{D})$, $Q \neq P$.

Proof. We first note that it suffices to define v_p at the vertex of each $\Delta \in \mathcal{D}$. Let $\mathcal{D} = \{\Delta^i\}$, $i = 1,\ldots,m$. Assume that we have denumerated the elements so that diam $\Delta^i \geq$ diam Δ^{i+1} $i = 1,\ldots,m-1$. We prove now our lemma by induction with respect to i.

First let us observe that all four vertices of Δ^1 are regular nodes. If one were irregular then by lemma 2.1, there exists $\tilde{\Delta} \in \mathcal{D}$ such that diam $\tilde{\Delta}$ > diam Δ^1. This is a contradiction because Δ^1 is the largest element.

Suppose now that $j = 1$, or $j > 1$ and v_p has already been constructed on $\bigcup_{i=1}^{i=j-1} \Delta^i$. Consider now the vertices of Δ^j. If Q is a vertex of Δ^j and

i) Q is a regular node, we define $v_p(Q) = 1$ for $Q = P$ and $v_p(Q) = 0$ for $Q \neq P$.

ii) Q is an irregular node, then by Lemma 2.1, $Q \in \Delta'$ where diam Δ' > diam Δ^j. By induction assumption v_p was already defined on Δ' and so $v_p(Q)$ is defined too. We therefore construct v_p on Δ^j with desired property on $\bigcup_{i=1}^{i=1} \Delta^i$. Let us remark that

if $Q \in \bigcup_{i=1}^{i=j-1} \Delta^i$ then the value $v_p(Q) = 1$ respectively 0 is the same as in the previous phase.

LEMMA 3.4. Let $\mathcal{D}$ be a mesh and $u \in M(\mathcal{D})$ be such that $u(Q) = 0$ for any $Q \in R(\mathcal{D})$ then $u = 0$.

Proof. Let $\{\Delta^i\}$ be numerated as in the proof of Lemma 3.3. The lemma will be proven by induction with respect to i. Because all vertices of Δ^1 are regular nodes we have $u = 0$ on Δ^1. Let now $i = 1,\ldots,j-1 < m$ and consider u on Δ^j. If the vertex P of Δ^j is a regular node then by assumption $u(P) = 0$. If P is irregular node then by lemma 2.1, we have also $P \in \Delta'$ with $\operatorname{diam} \Delta' > \operatorname{diam} \Delta^j$, i.e., $\Delta' = \Delta^k$, for some $k < j$. By induction $u = 0$ on $\bigcup_{i=1}^{i=j-1} \Delta^i$ and therefore $u(P) = 0$. So $u = 0$ in all vertices of Δ^j and so $u = 0$ on Δ^j and lemma is proven.

Lemmas 3.3 and 3.4 show that the function $u \in M(\mathcal{D})$ is uniquely defined by its values at the regular nodal points.

Lemma 3.3 and arguments analogous to those used in the proof of Lemma 3.3 yield

LEMMA 3.5. Let $\mathcal{D}$ be a mesh. Then

i) The set of functions $\{v_p | P \in R(\mathcal{D})\}$ creates a basis for $M(\mathcal{D})$

ii) $v_p \geq 0$

iii) $\sum_{P \in R(\mathcal{D})} v_p = 1$.

Definition 3.1: The set $\omega_p = \operatorname{supp} v_p$, $P \in R(\mathcal{D})$ will be called a star associated to the node P or briefly a P-star.

Lemma 3.5 (iii) yields readily that $\bigcup_{P \in R(\mathcal{D})} \omega_p = \bar{\Omega}$.

LEMMA 3.6. Let $\Delta \in \mathcal{D}$, $P \in R(\mathcal{D})$ and $\Delta \cap \operatorname{int} \omega_p \neq \emptyset$. Then $\Delta \subset \omega_p$.

Proof. Assume on the contrary that $\Delta \not\subset \omega_p$. Then there exists an open set $S \subset \Delta$, $S \cap \omega_p = \emptyset$. Because $v_p \in M(\mathcal{D})$, v_p is bilinear on Δ . But $v_p = 0$ on S , and hence $v_p = 0$ on Δ . This is a contradiction because we assumed that $\Delta \cap \operatorname{int} \omega_p \neq \emptyset$.

LEMMA 3.7. The set ω_p is connected in the sense that for any two elements $\Delta', \Delta'' \subset \omega_p$ there exists a sequence of elements $\Delta' = \Delta_0, \Delta_1, \ldots \Delta_n = \Delta''$ such that

i) $\Delta_i \subset \omega_p$

ii) $\Delta_i \cap \Delta_{i+1} \neq \emptyset$ and is the edge either Δ_i or Δ_{i+1}

Proof. Let $\omega_p = \bigcup_{j=1}^{s} \Delta^j$, $\Delta^j \in \mathcal{D}$. Assume that we have enumerated the elements Δ^j of ω_p so that diam $\Delta^j \geq$ diam Δ^{j+1} , $j = 1,\ldots,s-j$. It is obvious that we can restrict ourself to the case when $\Delta' = \Delta^1$.

First we prove that one of the vertices of Δ^1 is the node P . Suppose that the node P is not a vertex of Δ^1 . Let Q be any vertex of Δ^1 . If Q is regular, then $v_p(Q) = 0$. If Q is irregular, then there exists $\Delta^* \in \mathcal{D}$ with $Q \in \Delta^*$ and diam $\Delta^* >$ diam Δ^1 . Thus $\Delta^* \cap \operatorname{int} \omega_p = \phi$ and so $v_p = 0$ on Δ^* by lemma 3.6 and it follows that $v_p(Q) = 0$, so $v_p = 0$ on Δ^1 which is a contradiction. Now we prove the lemma by introduction with respect to j . Assume therefore that we are able to connect Δ^k with Δ^1 and consider the element Δ^{k+1} . If a vertex of Δ^{k+1} is the node P , then Δ^{k+1} can be connected with Δ^1 because Ω is a Lipschitz domain. If all vertices of Δ^{k+1} were regular nodes different from P then $\Delta^{k+1} \not\subset \omega_p$ by lemma 3.6.

So we need only consider the case where a vertex R of Δ^{k+1} is irregular

and $v_p(R) \neq 0$. By lemma 3.1 there exists $\Delta^* \in \mathcal{D}$ such that $\operatorname{diam} \Delta^* > \operatorname{diam} \Delta^{k+1}$, $\Delta^* \cap \Delta^{k+1}$ is an edge of Δ^{k+1} and $R \in \Delta^*$. Thus $\Delta^* \subset \omega_p$ and so $\Delta^* = \Delta^j$, $j \leq k$. So Δ^{k+1} can be connected with Δ^j and therefore with Δ^1 and lemma is proven.

Lemma 3.7 shows that $\operatorname{int} \omega_p$ is a domain.

So far we have not made any restrictions concerning the mesh $\mathcal{D}$. In the next section we will analyze the family of K-meshes, which play an essential role in the theory.

4. THE K-MESH

Definition 4.1. Let $K > 0$, real. A mesh $\mathcal{D}$ will be called a K-mesh if for any $P \in R(\mathcal{D})$

$$(4.1) \qquad \operatorname{diam} \omega_p \leq K \inf_{\substack{\Delta' \in \mathcal{D} \\ \Delta' \subset \omega_p}} \operatorname{diam} \Delta' \quad .$$

The definition has a clear sense because of Lemma 3.6.

We conjecture that definition 4.1 is equivalent to $\exists K^* > 0$ such that for all $\Delta \in \mathcal{D}$,
$$\sup_{\Delta \cap \Delta' \text{ is an edge of } \Delta'} \frac{\operatorname{diam} \Delta}{\operatorname{diam} \Delta'} \leq K^* \quad .$$

Everywhere in what will follow we shall assume that we deal only with K-meshes. We mostly will not mention it explicitly.

LEMMA 4.1. Suppose $\mathcal{D}$ is a K-mesh. Then there exists numbers M, N depending only on K such that

i) If $P \in R(\mathcal{D})$ then the P-star consists at most of N different elements of $\mathcal{D}$.

ii) If $\Delta' \in \mathcal{D}$ then $\Delta' \subset \omega_p$ for at most M different $P \in R(\mathcal{D})$.

iii) If $\Delta' \in \mathcal{D}$ then there are at most $4K + 4$ elements $\Delta'' \in \mathcal{D}$ such that $\Delta' \cap \Delta'' \neq \emptyset$.

Proof.

i) The star ω_p can be contained in a square S_p with its side $\operatorname{diam} \omega_p$. So for the number N_p of elements contained in ω_p we have the simple estimate

$$N_p \leq \frac{\operatorname{area} \omega_p}{\inf\limits_{\Delta' \in \omega_p} (\operatorname{area} \Delta')} \leq \frac{[\operatorname{diam} \omega_p]^2}{[\inf\limits_{\Delta' \subset \omega_p} (\operatorname{diam} \Delta')]^2} \leq K^2 \quad .$$

Hence $N < K^2$.

ii) If $\Delta' \subset \omega_p$ then $\operatorname{diam} \omega_p \leq K \operatorname{diam} (\Delta')$. Hence $\omega_p \subset Q_p$ where Q_p is a square with the center in the middle of Δ' and of the diameter $2(K-1/2) \operatorname{diam} \Delta'$. See Figure 4.1.

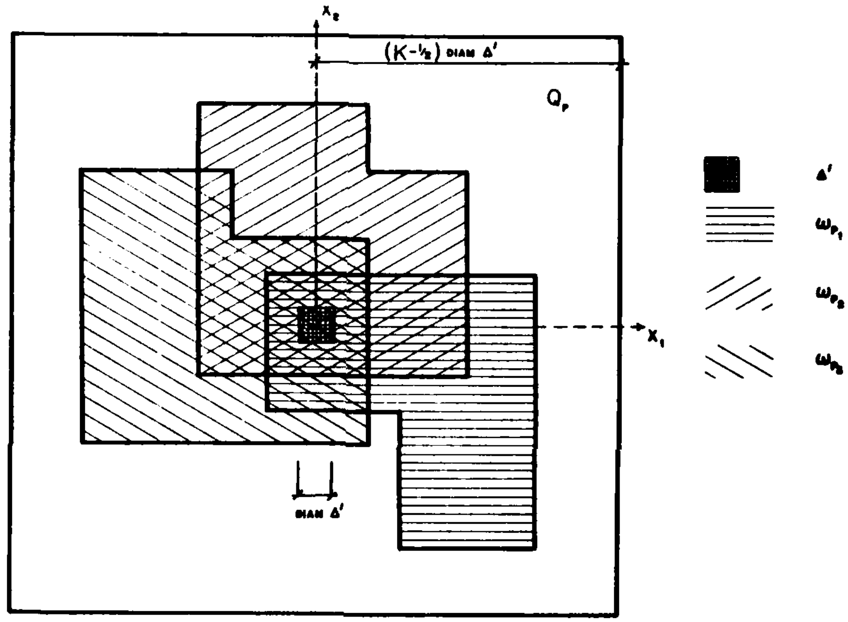


Figure 4.1. The Relation Between Δ' and ω_p .

Since $\Delta' \subset \omega_p$ we have

$$\operatorname{diam} \Delta' \leq \operatorname{diam} \omega_p \leq K \inf_{\Delta'' \subset \omega_p} \operatorname{diam} \Delta''$$

and hence for any $\Delta'' \subset \omega_p$

$$(4.2) \qquad \operatorname{diam} \Delta'' \geq \frac{\operatorname{diam} \Delta'}{K} .$$

The node P must be a vertex of some $\Delta'' \subset \omega_p$ satisfying (4.2). The number of such elements is obviously bounded by

$$\frac{\text{area } Q_p}{\left(\frac{\text{diam } \Delta'}{K}\right)} \le 4K^2(K-1/2)^2 \quad .$$

Because not more than 4 regular nodes could be on any element, we see that

$$M \le 16K^2(K-1/2)^2 \quad .$$

iii) Any vertex of Δ' can be a vertex of at most three other elements and therefore by lemma 3.1 it is sufficient to bound the number of elements Δ'' with diam $\Delta'' \le$ diam Δ' such that $\Delta'' \cap \Delta'$ is an edge of Δ'' contained in some edge of Δ', say Γ. Fix this Γ, and suppose that there are q such Δ''. q is finite since by the definition of $\mathcal{D}$, there is only a finite number of elements. Then for at least one such Δ'', $\tilde{\Delta}$ say, diam $\tilde{\Delta} \le \frac{1}{q}$ diam Δ'. Since $\sum_{P \in R(\mathcal{D})} v_p = 1$ by lemma 3.5, there exists $P \in R(\mathcal{D})$ such that v_p is not identically zero on $\tilde{\Delta} \cap \Delta'$, and therefore both $\tilde{\Delta}$ and $\Delta' \subset \omega_p$. But

$$q \text{ diam } \tilde{\Delta} \le \text{ diam } \Delta' \le \text{ diam } \omega_p \le K \text{ diam } \tilde{\Delta} \quad .$$

Hence

$$q \le K$$

and the lemma easily follows.

<u>LEMMA 4.2.</u> <u>There exists a number L (depending only on K) such that</u>

$R(\mathcal{D})$ can be partitioned into $j \leq L$ sets χ_j, $j = 1,\ldots,L$ such that if $P,Q \in \chi_j$, $P \neq Q$ then $\operatorname{int}\omega_P \cap \operatorname{int}\omega_Q = \emptyset$.

Proof. Suppose $P \neq Q$, $P,Q \in R(\mathcal{D})$ and $\operatorname{int}\omega_P \cap \operatorname{int}\omega_Q \neq \emptyset$. By lemma 3.6 $\operatorname{int}\omega_P \cap \operatorname{int}\omega_Q$ must contain the entire interior of at least one element. By Lemma 4.1 ω_P contains at most N elements and each of these elements can be contained in $\omega_{P'}$ for at most $M-1$ nodes $P' \in R(\mathcal{D})$, $P \neq P'$. Therefore there can be at most $N(M-1)$ regular nodes Q such that $\operatorname{int}\omega_P \cap \operatorname{int}\omega_Q \neq \emptyset$.

We shall construct now the sets χ_j by the following recursive procedure. Let $P_1, P_2,\ldots,P_r$ be some enumeration of the regular nodes. Set $\chi_1^1 = \{P_1\}$. Suppose that we have already defined sets $\chi_1^{\ell-1},\ldots,\chi_{S_{\ell-1}}^{\ell-1}$ for some $S_{\ell-1} \geq 1$, $\ell \leq r - 1$. If for $P_\ell \in R(\mathcal{D})$ and some $1 \leq k \leq S_{\ell-1}$, $\operatorname{int}\omega_{P_\ell} \cap \operatorname{int}\omega_Q = \emptyset$ for all $Q \in \chi_k^{\ell-1}$, then choose k to be minimal, and set

$$\chi_t^\ell = \chi_t^{\ell-1} \quad \text{for} \quad t \in \{1,\ldots,S_{\ell-1}\} - \{k\}$$

$$\chi_k^\ell = \chi_k^{\ell-1} \cup \{P_\ell\}$$

and define $S_\ell = S_{\ell-1}$. Otherwise set

$$\chi_t^\ell = \chi_\ell^{\ell-1} \quad \text{for} \quad 1 \leq t \leq S_{\ell-1}$$

$$\chi_{S_{\ell-1}+1} = \{P_\ell\}$$

with $S_\ell = S_{\ell-1} + 1$.

Now from the first part of our proof we see that $S_j \leq N(M-1) + 1$ and so $L \leq N(M-1) + 1$ and the lemma is proven.

Remark. As we mentioned in the introduction, this paper develops the basic ideas of [1] . The lemma 4.1 relates to the intersection index and the lemma 4.2 to the overlap index as introduced there.

Given an element $\Delta \in \mathcal{D}$ we will always enumerate its vertices as shown in Figure 4.2.

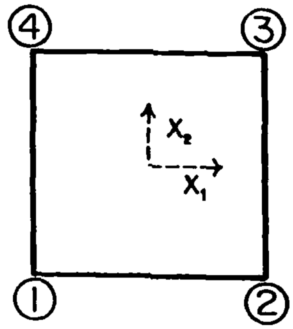


Figure 4.2. The Numeration of the Vertices

Let $P \in \mathcal{R}(\mathcal{D})$ and ω_p its star. The node P is a vertex of at most 4 elements. We will denote by $\Delta_p \subset \omega_p$ the element for which the vertex number given by Fig. 4.2 has the minimal value. This rule associates to every P a unique element.

Let now ω_p be a P-star. Then by $\mathcal{J}_p$ we denote the invertible affine transformation taking P to the origin and $\mathcal{J}_p(\Delta_p) = [0,1] \times [1,0]$. Further let $\Phi_p = \mathcal{J}_p(\omega_p)$. Φ_p will be called a standard P-star and we shall call members of $\mathcal{S}_p = \{\mathcal{J}_p(\Delta) \mid \Delta \in \mathcal{D} ,\ \Delta \subset \omega_p\}$ the standard P elements of Φ_p and if no confusion arises denote them also by Δ . Note that Φ_p can equal Φ_Q for $P \neq Q$ and yet $\mathcal{S}_p \neq \mathcal{S}_Q$.

LEMMA 4.3.

i) int Φ_p is a domain.

ii) There are not more than $Z = A(K)$ possibilities for $\mathcal{S}_p$ as P ranges over $\mathcal{R}(\mathcal{D})$.

Proof.

i) Follows immediately by lemma 3.7.

ii) The result will follow if we can show that there is a finite collection of squares $S_1,\ldots,S_{\pi(k)}$ such that $\{J_p(\Delta) \mid \Delta \in \mathcal{D},\ \Delta \subset \omega_p\} = \{S_1,\ldots S_{\pi(k)}\}$.

To this end we show by induction that the vertices of each $J_p(\Delta)$ must have coordinates of the form $(k\eta, \ell\eta)$ where $\eta = 2^{-([\ell g_2 K]^{int}+1)}$ k,ℓ integral $|k|,|\ell| \leq 2K^2$ and that $\operatorname{diam} J_p(\Delta) = 2^{\lambda}\eta$ for some $\lambda = 1,2,3,\ldots$ Here $[\cdot]^{int}$ denotes the integral part.

Now if $\Delta \in \mathcal{D}$, $\Delta \subset \omega_p$ we can construct a sequence

$$\Delta_p = \Delta_o,\ldots,\Delta_n = \Delta$$

having the properties mentioned in Lemma 3.7.

Clearly any vertex of $J(\Delta_p)$ satisfies the above inductive hypothesis. So suppose it holds for $J(\Delta_o),\ldots,J(\Delta_i)$ $0 \leq i \leq n-1$. Then by Lemma 3.1 (3) $\operatorname{diam} \Delta_{i+1} = 2^s \operatorname{diam} \Delta_i$, s integral, giving $\operatorname{diam} J_p(\Delta_{i+1}) = 2^s \operatorname{diam} J_p(\Delta_i) = 2^{\lambda+s}\eta$ for some integral λ, s . Since $\Delta_{i+1} \subset \omega_p$ we have as in 4.2 $\operatorname{diam} \Delta_{i+1} \geq \operatorname{diam} \Delta_p/K$. This gives $\operatorname{diam}(J_p(\Delta_{i+1})) \geq \frac{1}{K}$ and so $\lambda + s > 0$ and so we conclude that $\operatorname{diam} J_p(\Delta_{i+1}) = 2^t\eta$ with $t \geq 1$ integral.

Suppose for the moment that $\Delta_{i+1} \leq \operatorname{diam} \Delta_i$. Appealing to Lemma 3.1 (1) we see that the two vertices of $J_p(\Delta_{i+1})$ which are the end points of $J_p(\Delta_{i+1}) \cap J_p(\Delta_i)$ must satisfy the induction hypotheses. Since $J_p(\Delta_{i+1})$ is a square, it follows also for the other two vertices. The bound on $|k|,|\ell|$ is a consequence of $\operatorname{diam} \Phi_p \leq 2K$. The case $\operatorname{diam} \Delta_i \leq \operatorname{diam} \Delta_{i+1}$ follows by a similar argument.

COROLLARY: There is not more than $Z(K)$ of different possible domains Φ_p .

Given $\Delta \in \mathcal{D}$ we denote

$$(4.3) \qquad Q^*(\Delta) = \bigcup \{\omega_p \mid P \in \mathcal{R}(\mathcal{D}) \ , \ \Delta \subset \omega_p\} \quad .$$

As in the proof of Lemma 4.1 we have $Q^*(\Delta) \subseteq Q(\Delta)$, where $Q(\Delta)$ is a square with the center at the middle of Δ and

$$(4.4) \qquad \operatorname{diam} Q(\Delta) \leq 2(K-1/2) \operatorname{diam} \Delta \quad .$$

LEMMA 4.4

i) If $\Delta', \Delta'' \in \mathcal{D}$ and $\Delta', \Delta'' \subset Q^*(\Delta)$ then

$$(4.5) \qquad \operatorname{diam} \Delta' > \frac{1}{K} \operatorname{diam} \Delta$$

and

$$(4.6) \qquad \operatorname{diam} \Delta' \geq \frac{1}{K^2} \operatorname{diam} \Delta'' \quad .$$

ii) Int $Q^*(\Delta)$ is a domain .

iii) If $P \in \mathcal{R}(\mathcal{D})$ and $\omega_p \subset Q^*(\Delta)$ then on Δ

$$(4.7) \qquad |D^1 v_p| \leq K(\operatorname{diam} \Delta)^{-1} \quad .$$

For any $\Delta \subset w_p$

$$(4.8) \qquad |D^2 v_p| \leq K^2 (\operatorname{diam} \Delta)^{-2} \quad \text{on} \quad \Delta \ .$$

Proof. Let $\Delta' \subset Q^*(\Delta)$ then for some $P' \in \mathcal{R}(\mathcal{D})$ we have $\Delta, \Delta' \in \omega_{p'}$, hence

$$(4.9) \qquad \text{diam } \Delta' \geq \frac{1}{K} \text{diam } \omega_{p'} \geq \frac{1}{K} \text{diam } \Delta \quad .$$

Further we have for $\Delta'' \subset \omega_{p''}$

$$(4.10) \qquad \text{diam } \Delta \geq \frac{1}{K} \text{diam } \omega_{p''} \geq \frac{1}{K} \text{diam } \Delta''$$

(4.9) and combination of (4.9) and (4.10) yield (4.5) and (4.6).

ii) We have to prove only that $Q^*(\Delta)$ is connected. This follows immediately from the Lemma 3.7.

iii) From Lemma 3.5 (ii) and (iii) we have $0 \leq v_p \leq 1$ and so obviously

$$|D^1 v_p| \leq [\min_{\Delta \subset \omega_p} \text{diam } \Delta]^{-1}$$

and

$$|D^2 v_p| \leq [\min_{\Delta \subset \omega_p} \text{diam } \Delta]^{-2} \quad .$$

(4.9) yields the Lemma.

Let $K \leq 2^{s_o}$. Denote

$$Q^{**}(\Delta) = \{x \in Q^*(\Delta) \,|\, \text{dist}(x, \partial Q^* - \partial\Omega) \geq \frac{\text{diam } \Delta}{2^{s_o+2}} \}$$

LEMMA 4.5.

i) Let $\omega_p \subset Q^*(\Delta)$. Then $P \in Q^{**}(\Delta)$.

ii) $\Delta \subset Q^{**}(\Delta)$

iii) int $Q^{**}(\Delta)$ is a domain.

Proof.

i) Assume that $P \notin Q^{**}(\Delta)$ then dist $(P, \partial Q^* - \partial\Omega) < \operatorname{diam} \Delta / 2^{s_o+2}$.

Because $v_p = 0$ on $\partial Q^* - \partial\Omega$, (4.7) leads obviously to a contradiction.

ii) Assume that $\Delta \not\subset Q^{**}(\Delta)$. Then there is a vertex P^* of Δ such that dist $(P^*, \partial Q^* - \partial\Omega) < (\operatorname{diam} \Delta) 2^{-(s_o+2)}$. Obviously $w = \sum_{\omega_p \subset Q^*(\Omega)} v_p = 1$ on Δ . By the same argument as leading to (4.7) we see that $|D^1 w| \leq K(\operatorname{diam} \Delta)^{-1}$. Because $w = 0$ on $\partial Q^* - \partial\Omega$ we have a contradiction.

iii) Let $x \in Q^{**}(\Delta)$, $x \in \Delta_o \subset \omega_p$. Then by Lemma 3.7 there exists sequence $\Delta_o, \Delta_1, \ldots \Delta_n = \Delta$ such that $V = \operatorname{int} \bigcup_{j=0}^{n} \Delta_j$ is a domain. Because $\operatorname{diam} \Delta_j \geq \frac{1}{K} \operatorname{diam} \Delta$ by (4.2) it is readily seen that int $Q^{**} \cap V$ is a domain. This leads immediately to the desired result.

Let $\psi_p^* = J_p(Q^*(\Delta))$ (analogously as Φ_p) and $\psi_p^{**} = J_p(Q^{**}(\Delta))$. Then we have

LEMMA 4.6. There exists not more than $Z^*(k)$ (respectively $Z^{**}(k)$) possible domains int ψ_p^* (respectively int ψ_p^{**}) .

Proof. The first part of the lemma follows from Lemma 4.3 and is a corallary. The second part follows by analogous argument when realizing that ψ_p^{**} is composed by squares with diam $\eta/4$ when η was introduced in the proof of Lemma 4.3.

5. THE APPROXIMATION PROPERTIES OF $M(\mathcal{D})$

In this section we will analyze the approximation properties of $M(\mathcal{D})$.

Let Π be the mapping of $C^o(\Omega)$ onto $M(\mathcal{D})$ such that $(\Pi u)(P) = u(P)$, $P \in R(\mathcal{D})$. By Lemma 3.5 we can write

$$\Pi u = \sum_{P \in R(\mathcal{D})} u(P) v_P \quad .$$

Further define the operator $\Pi^{Q^*(\Delta)}$ mapping $C^o(\Omega)$ into $M(\mathcal{D})$ by

$$(5.1) \qquad \Pi^{Q^*(\Delta)}(u) = \sum_{\substack{P \in R(\mathcal{D}) \\ \omega_P \subset Q^*(\Delta)}} u(P) v_P$$

Clearly $\operatorname{supp}(\Pi^{Q^*(u)}(u)) \subset Q^*(\Delta)$ and

$$(5.2) \qquad \Pi^{Q^*(\Delta)}(u) = \Pi(u) \quad \text{on} \quad \Delta \quad .$$

For given $\Delta \in \mathcal{D}$ we define J_Δ as the invertible affine transformation of $\mathbb{R}^2$ onto $\mathbb{R}^2$ taking Δ into standard unit square $[0,1] \times [0,1] = S$.

LEMMA 5.1. There exists a constant C dependent only on K such that for any $w \in H^2(\Omega)$ and $\ell = 0,1$

$$(5.3) \qquad \| w o J_\Delta^{-1} - (\Pi^{Q^*(\Delta)} w) o J_\Delta^{-1} \|_{H^\ell(S)} \le C \| w o J_\Delta^{-1} \|_{H^2(J_\Delta Q^{**}(\Delta))}$$

Proof. By Sobolev imbedding theorem we have

$$(5.4) \qquad \|w \circ J_\Delta^{-1}\|_{C^0(J_\Delta Q^{**}(\Delta))} \le C\|w \circ J_\Delta^{-1}\|_{H^2(J_\Delta Q^{**}(\Delta))} .$$

Because of Lemma 4.6 there are at most $Z^{**}(K)$ different domains $J_\Delta Q^{**}(\Delta)$, where $Z^{**}(K)$ depends on K only. This shows that C in (5.4) depends on K only.

Using (5.1), Lemma 4.4 and Lemma 4.5 i) we get

$$(5.5) \qquad \|(\Pi^{Q^*(\Delta)} w) \circ J_\Delta^{-1}\|_{H^\ell(S)} \le C\|w \circ J_\Delta^{-1}\|_{H^2(J_\Delta Q^{**}(\Delta))}$$

where C depends on K only.

Since by Lemma 4.5 $\Delta \subset Q^{**}(\Delta)$

$$(5.6) \qquad \|w \circ J_\Delta^{-1}\|_{H^\ell(s)} \le \|w \circ J_\Delta^{-1}\|_{H^2(J_\Delta Q^{**}(\Delta))} .$$

Combining (5.5) and (5.6) we get the Lemma.

LEMMA 5.2. There exists a constant C (dependent only on K) such that for any $w \in H^2(\Omega)$ and $\ell = 0,1$

$$(5.7) \qquad |w \circ J_\Delta^{-1} - (\Pi^{Q^*(\Delta)} w) \circ J_\Delta^{-1}|_{H^\ell(s)} \le C|w \circ J_\Delta^{-1}|_{H^2(J_\Delta Q^{**}(\Delta))}$$

Proof. Suppose z is any function bilinear on Q^{**} . Then on Δ we have $\Pi^{Q^*(\Delta)} z = z$ and therefore

$$(w+z)\circ J_\Delta^{-1} - (\Pi^{Q^*(\Delta)}(w+z))\circ J_\Delta^{-1} = w\circ J_\Delta^{-1} - (\Pi^{Q^*(\Delta)}w)\circ J_\Delta^{-1}$$

and hence using lemma 5.1 we get

$$||w\circ J_\Delta^{-1} - (\Pi^{Q^*(\Delta)}w)\circ J_\Delta^{-1}||_{H^\ell(s)} \leq C \inf_z ||w\circ J_\Delta^{-1} - z||_{H^2(J_\Delta Q^{**}(\Delta))}$$

where z is an arbitrary bilinear function on $J_\Delta Q^{**}(\Delta)$.

Because by lemma 4.6 there is only a finite number (depending on K only) of different $J_\Delta Q^{**}(\Delta)$, and by lemma 4.5 any int $J_\Delta Q^{**}(\Delta)$ is a domain, we have (see e.g. [2], pp.) that

$$\mathrm{Inf}\, ||w\circ J^{-1} - z||_{H^2(J_\Delta Q^{**}(\Delta))} \leq C|w\circ J^{-1}|_{H^2(J_\Delta Q^{**}(\Delta))} .$$

The theorem follows immediately.

Now we have

THEOREM 5.3. There exists a constant C depending only on K such that for any $\Delta \in \mathcal{D}$, $u \in H^2(\Omega)$ and $\ell = 0,1$

$$(5.8) \qquad ||u - \Pi u||_{H^\ell(\Delta)} \leq C(\mathrm{diam}\, \Delta)^{2-\ell} ||u||_{H^2(Q^{**}(\Delta))} .$$

Proof. Using (5.2) we obtain (5.8) from lemma 5.2 by standard scaling argument.

THEOREM 5.4. Let $u \in H^2(\Omega)$ then for $\ell = 0,1$

(5.9) $\underline{||u-\Pi u||_{H^{\ell}(\Omega)} \leq Ch^{2-\ell}(\mathcal{D})||w||_{H^2(\Omega)}}$

<u>with C depending only on K .</u>

Proof. We have

$$||u-\Pi u||^2_{H^{\ell}(\Omega)} = \sum_{\Delta\in\mathcal{D}} ||u-\Pi u||^2_{H^{\ell}(\Delta)} \leq C\sum_{\Delta\in\mathcal{D}} ||u||^2_{H^{\ell}(Q^{**}(\Delta))} h^{2(2-\ell)}(\mathcal{D}) \leq$$

$$\leq C\sum_{\Delta\in\mathcal{D}} ||u||^2_{H^2(Q^*(\Delta))} h^{2(2-\ell)}(\mathcal{D}) \leq C\sum_{\Delta\in\mathcal{D}} \lambda(\Delta)||u||^2_{H^2(\Delta)} h^{2(2-\ell)}(\mathcal{D})$$

where $\lambda(\Delta)$ is the number of $Q^*(\Delta')$ such that $\Delta\subset Q^*(\Delta')$. Using lemma 4.1 we see that $\lambda(\Delta) \leq MN \leq C(k)$ and so theorem 5.4 is proven.

Remark: In Theorems 5.3 and 5.4, the restriction $u\in H^2(\Omega)$ can be weakened to $u\in H^2(\Delta)$ for every $\Delta\in\mathcal{D}$.

Theorem 5.4 shows that $M(\mathcal{D})$ has the same basic "interpolation" properties as the usual finite element spaces. The spaces $M(\mathcal{D})$ are more flexible than the usual spaces defined on quadrilateral meshes. The space $M(\mathcal{D})$ allows us to make a refinement and still keep square elements. The restriction to K-meshes is from a practical point not important. A more essential restriction is that we deal only with squares. How to overcome this resctiction with respect to the implementation and the theory will be discussed elsewhere.

The use of the spaces $M(\mathcal{D})$ does seem to have a major advantage over triangular elements because of programming and data management simplicity, expecially when some form of automatic or adaptive mesh generator is envisaged. One manifestation of this is that the element can be uniquely defined by the binary

expansion of the coordinates of its center, the length of such an expansion indicating the size of the element. Since all the elements have moduli, a scaling factor the same geometric shape, the calculation of the stiffness matrix, etc. is simplified. In addition this seems to be important for the practical effectiveness of the error estimation.

We shall analyze now the approximation properties of $M(\mathcal{D})$ when $u \in H^1(\Omega)$.

Before being that, we introduce some notations. By $\mu(x)$, $x \in \mathbb{R}^2$ we denote a molifier, a function with all derivatives continuous $\mu(x) \geq 0$, $\mu(x) = 0$ for $|x| \geq 1$, $\mu(0) = 1$, and $\int_{\mathbb{R}_2} \mu(x)dx = A$. For $\varepsilon > 0$, let $\mu_\varepsilon(x) = \frac{1}{\varepsilon^2 A}\mu(x/\varepsilon)$.

Let $\Omega \subset \mathbb{R}^2$ be an arbitrary bounded domain and Ω^ρ its ρ-neighborhood. For u defined on Ω^ρ, we put

$$u_\varepsilon = u * \mu_\varepsilon \quad , \quad \varepsilon < \rho \quad .$$

Obviously u_ε is defined on Ω.

Further for any $t \in \mathbb{R}^2$, $|t| \leq 1$, $\varepsilon < \rho$ let

$$u^{t\varepsilon}(x) = u(x+t\varepsilon) \quad .$$

Then $u^{t\varepsilon}$ is also defined on Ω, and we have

(5.10) $$||u_\varepsilon||_{H^2(\Omega)} \leq C\varepsilon^{-1}||u||_{H^1(\Omega^\rho)}$$

$$||u_\varepsilon - u||_{H^\ell(\Omega)} \leq C\varepsilon^{1-\ell}||u||_{H^1(\Omega^\rho)} \quad , \quad \ell = 0,1$$

$$||u^{t\varepsilon}-u||_{H^{\ell}(\Omega)} \leq C\varepsilon^{1-\ell}||u||_{H^{1}(\Omega^{\rho})}$$

with C independent on Ω , u and ε .

Let us prove now

LEMMA 5.5. For every $P \in R(\mathcal{D})$, let a function $w_p \in H^2(\Omega)$ be given. Let $\Delta \in \mathcal{D}$ and $\Delta \subset \omega_{p_o}$. If

$$W = \Pi \sum_{P \in R(\mathcal{D})} v_p w_p$$

then for $\ell = 0,1$

$$(5.11) \qquad ||w_{p_o} - W||_{H^{\ell}(\Delta)} \leq C[(\operatorname{diam} \Delta)^{2-\ell}||w_{p_o}||_{H^2(Q^{**}(\Delta))} +$$

$$+ \sum_{\substack{P \in R(\mathcal{D}) \\ \Delta \in \omega_p}} \sum_{i=0}^{2} ||-w_{p_o} + w_p||_{H^i(Q^{**}(\Delta))} (\operatorname{diam} \Delta)^{i-\ell}] .$$

Proof. On Δ we have

$$W = \Pi \sum_{P \in R(\Delta)} v_p w_{p_o} + \Pi \sum_{P \in R(\Delta)} v_p[-w_{p_o} + w_p]$$

$$= \Pi^{Q^*(\Delta)} w_{p_o} + \Pi^{Q^*(\Delta)} \sum_{P \in R(\Delta)} v_p[-w_{p_o} + w_p] .$$

Using lemma 4.4 and Leibnitz's rule, we have for $\ell = 0,1,2$

$$(5.12) \qquad \sum_{\Lambda' \cap Q^{**}(\Lambda) \neq \emptyset} \| v_p [-w_{p_o} + w_p] \|_{H^\ell(Q^{**}(\Lambda) \cap \Lambda')} \leq$$

$$\leq C \sum_{i=0}^{\ell} \| -w_{p_o} + w_p \|_{H^i(Q^{**}(\Delta))} (\operatorname{diam} \Lambda)^{i-\ell}$$

Therefore on Δ

$$w_{p_o} - W = w_{p_o} - \Pi^{Q^*(\Lambda)} w_{p_o} - \sum_{P \in R(\Delta)} v_p[-w_{p_o} + w_p] - [\Pi^{Q^*(\Delta)}(\sum_{P \in R(\Delta)} v_p[-w_{p_o} + w_p]$$

$$- \sum_{P \in R(\Delta)} v_p[-w_{p_o} + w_p]) \ .$$

Applying theorem 5.3 and noting the remark after theorem 5.4, we get the lemma.

LEMMA 5.6. Let $v \in H^1(\mathbb{R}^2)$. Associate to every $P \in R(\mathcal{D})$ a sector $t_p \in \mathbb{R}^2$, $|t_p| \leq 1$ and a number λ_p such that

$$\lambda_p = \left[\min_{\Lambda \in \cdot_p} (\operatorname{diam} \Lambda) \right] \frac{1}{K.512} \ .$$

Let further

$$w_p = \mu_{\frac{\lambda_p}{64}} * [v^{t_p \lambda_p}]$$

and

$$W = \Pi \sum_{P \in R(\mathcal{D})} v_p w_p \ .$$

Then for any $\Delta \in \mathcal{D}$ and $\ell = 0,1$

$$||v-W||_{H^{\ell}(\Delta)} \leq C(\text{diam } \Delta)^{1-\ell} ||v||_{H^1(Q^{**}(\Delta))^{2\rho}} ,$$

$$(5.13) \qquad \rho = \max_{\omega_p \subset Q^*(\Delta)} \lambda_p \quad .$$

Proof. Let $\Delta \subset \omega_{p_o}$ and $\Delta \subset \omega_p$. Then using lemma 4.4 we have $\lambda_p, \lambda_{p_o} \leq C(K)$ diam Δ. Using (5.10) we get for $i = 0,1,2$

$$||w_p - w_{p_o}||_{H^i(Q^{**}(\Delta))} \leq C(\text{diam } \Delta)^{1-i} ||v||_{H^1([Q^{**}(\Delta)]^{2\rho})} \quad .$$

By lemma 5.5 we get readily the lemma.

LEMMA 5.7. Let ρ be defined by (5.13), then

$$[Q^{**}(\Delta)]^{4\rho} \cap \Omega \subset Q^*(\Delta) \quad .$$

The lemma follows easily from the definition.

THEOREM 5.8. Let $u \in H^1(\Omega)$ respectively $H^{1,\Gamma}(\Omega)$. Then for any $\mathcal{D}$ with $h(\mathcal{D}) \leq \varepsilon_o$ there exists $w \in M(\mathcal{D})$ respectively $M(\mathcal{D}) \cap H^{1,\Gamma}(\Omega)$ such that for all $\Delta \in \mathcal{D}$ and $\ell = 0,1$

$$(5.14) \qquad ||u-w||_{H^{\ell}(\Delta)} \leq C(\text{diam } \Delta)^{1-\ell} ||u||_{H^1(Q^*(\Delta))}$$

with C independent of $\mathcal{D}$, u and Δ .

Proof. The extension of u onto a neighborhood Ω^{ρ} , has the properties listed in theorem 2.1. Denote by $[Q^{**}(\Delta)]^{2\rho_o}$ the $2\rho_o$-neighborhood of $Q^{**}(\Delta)$ $\rho_o = \min[\rho, \rho/2\beta]$ where ρ is defined by (5.13) and β was introduced in Theorem 2.1. Then by lemma 5.7 and theorem 2.1

$$||u||_{H^1([Q^{**}(\Delta)]^{2\rho})} \leq C||u||_{H^1([Q^{**}(\Delta)]^{4\rho} \cap \Omega)} \leq C||u||_{H^1[Q^{*}(\Delta)]} .$$

Let $u \in H^1(\Omega)$. Select $t_p = 0$ in Lemma 5.6 for all $P \in R(\mathcal{D})$. Then for W constructed in Lemma 5.6 we have $W \in M(\mathcal{D})$ and

$$||u-W||_{H^1(\Delta)} \leq C||u||_{H^1[(Q^{**}(\Delta)^{2\rho}]} \leq C||u||_{H^1[Q^{*}(\Delta)]}$$

and the first part of the theorem is proven.

We have now to show that when $u \in H^{1,\Gamma}(\Omega)$ then we can choose W so that $W(P) = 0$ for all $P \in R(\mathcal{D})$, $P \in \Gamma$. To every $P \in R(\mathcal{D})$, $P \notin \Gamma$, we take $t_p = 0$. If $P \in \partial\Omega$, $P \in \Gamma$, $P \in \partial\Gamma$ then we take t_p the outward unit normal (if P is a corner of Ω then the normal is bisecting the outside angle). If $P \in \partial\Gamma$ then it is easy to see that we can select vector t_p (pointing in to the sector Ω^{ρ}_{**} in Fig. 2.1) and not necessarily of the unit length such that $w_p(P) = 0$ also. This finishes the proof.

Finally we prove

THEOREM 5.9. Let $u \in H^1(\Omega)$ [respectively $\mathring{H}^1(\Omega)$, $H^{1,\Gamma}(\Omega))$]. Then there exists $w \in M(\mathcal{D})$ respectively $M(\mathcal{D}) \cap \mathring{H}^1(\Omega)$ respectively $M(\mathcal{D}) \cap H^{1,\Gamma}(\Omega)$ such that

(5.15) $\underline{\sum_{P\in R(\mathcal{D})} ||v_p(u-w)||^2_{H^1(\Omega)} \le C||u||^2_{H^1(\Omega)}}$

<u>with the constant C dependent only on K</u> .

<u>Proof</u>. By Theorem 5.8 we see that for any $\Delta\in\mathcal{D}$ and $\ell = 0,1$

(5.16) $$||u-w||_{H^\ell(\Delta)} \le C(\mathrm{diam})^{1-\ell}||u||_{H^1(Q^*(\Delta))}$$

and applying Leibnitz's rule and lemma 4.4, we obtain

(5.17) $$||v_p(u-w)||_{H^1(\Delta)} \le C||u||_{H^1(Q^*(\Delta))}$$

for all $P\in R(\mathcal{D})$ such that $\omega_p \subset Q^*(\Delta)$. By lemma 4.1 there are not more than $M(K)$ such nodes so we have

$$\sum_{P\in R(\mathcal{D})} ||v_p(u-w)||^2_{H^1(\Delta)} \le \sum_{\substack{P\in R(\mathcal{D}) \\ \omega_p \subset Q^*(\Delta)}} ||v_p(u-w)||^2_{H^1(\Delta)} \le C(K)||u||^2_{H^1(Q^*(\Delta))}$$

and therefore

(5.18) $$\sum_{P\in R(\mathcal{D})} ||v_p(u-w)||^2_{H^1(\Omega)} \le C(K)\sum_{\Delta\in\mathcal{D}} ||u||^2_{H^1(Q^*(\Delta))} .$$

By lemma 4.1 there are not more than $C(K)$ different $\Delta'\in\mathcal{D}$ such that $Q^*(\Delta')\supset\Delta$ and so (5.18) yields (5.15).

Remark to Theorem 5.9. *Assume now that* $\operatorname{int} \operatorname{supp} u = \Omega^*$. *Then*

$$\operatorname{supp} w = \bigcup_{\substack{P \in \mathcal{R}(\mathcal{D}) \\ \omega_P \cap \Omega^* \neq \emptyset}} \omega_P \quad .$$

This observation follows immediately from (5.16).

Remark: Theorem 5.9 is closely related to (3.1) of [9] which is an essential part of the a-posteriori error analysis.

6. THE MODEL PROBLEM, ITS FINITE ELEMENT SOLUTION AND THE BASIC A-POSTERIORI ESTIMATE

As a model problem we will discuss the case of plane elasticity for a body, homogeneous and isotropic on every Q_θ^k making up the domain Ω.

Let $\overset{\circ}{H}{}^1(\Omega) \subset H_i \subset H^1(\Omega)$ $i = 1,2$ where $H_i = H^{1,\Gamma_i}(\Omega)$ as defined in Section 2.

Let $H_0 = H_1 \times H_2$, with $u = (u_1,u_2)$ and consider on $H_0 \times H_0$ the following bilinear form

$$(6.1) \qquad B(u,v) = B(u_1,u_2;v_1,v_2) = \iint_\Omega \Big[(\lambda+2\mu)\Big(\frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2}\Big) + $$

$$+ \mu\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big)\Big(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\Big) + \lambda\Big(\frac{\partial u_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_1}{\partial x_1}\Big)\Big]dx \quad .$$

We assume that λ and μ are positive and constant on every Q^k of Ω. The constants λ, μ are the usual Lamé constants.

We will assume that there exist constants $0 < C_1, C_2 < \infty$ such that for any $u \in H_0$

$$(6.2) \qquad C_1 ||u||_{H^1(\Omega)} \le B(u,u) \le C_2 ||u||_{H^1(\Omega)}$$

with

$$||u||^2_{H^1(\Omega)} = ||u_1||^2_{H^1(\Omega)} + ||u_2||^2_{H^1(\Omega)} \quad .$$

Obviously $B(u,v) = B(v,u)$ and $|B(u,v)| \leq C||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)}$. Therefore on H_0, $B(u,v)$ is a scalar product with the energy norm

$$(6.3) \qquad |||u|||^2 = B(u,u) \quad .$$

(6.2) shows that the energy norm $|||u|||^2$ is equivalent with $||u||_{H^1(\Omega)}$.

The problem $\mathcal{P}(H_0,w,g)$, $w \in H^1(\Omega) \times H^1(\Omega)$, $g \in H^0(\Omega) \times H^0(\Omega)$ consists of finding $u \in H^1(\Omega) \times H^1(\Omega)$ such that $u - w \in H_0$ and

$$(6.4) \qquad B(u,v) = (g,v), \quad \forall v \in H_0$$

where we have written

$$(g,v) = \int_\Omega (g_1 v_1 + g_2 v_2)dx \quad .$$

It follows by the standard theory that u exists and is uniquely determined. The function u will be called the exact solution of the problem $\mathcal{P}$.

Let now $M_i(\mathcal{D}) = M(\mathcal{D}) \cap H_i(\Omega)$, $i = 1,2$ and $M_0(\mathcal{D}) = M_1(\mathcal{D}) \times M_2(\mathcal{D})$. Assuming that $w \in [M(\mathcal{D})]^2$ in the problem $\mathcal{P}(H_0,w,g)$, the finite element solution U of $\mathcal{P}$ with respect to M consists of finding $U \in M(\mathcal{D}) \times M(\mathcal{D})$ such that $U-w \in M_0(\mathcal{D})$ and

$$(6.5) \qquad B(U,v) = (g,v), \quad \forall v \in M_0(\mathcal{D}) \quad .$$

Just as for the exact solution it follows that the finite element solution U exists and is uniquely determined.

Finally we denote by $e = U - u$ the <u>error of the finite element solution</u>. We will be interested in an a-posteriori estimate of the norm $||e||_{H^1}$, $|||e|||$ or some norm equivalent to it. We will design an estimator E -- depending only on the known finite element solution -- which will be related to the error norm. As estimator E is called an <u>upper</u> respective to <u>lower</u> <u>estimator</u> if there exist constants A_U respectively A_L independent of $\mathcal{D}$ and u, U such that

$$||e|| \leq A_U E$$

respective

$$A_L E \leq ||e|| \qquad .$$

<u>THEOREM 6.1</u>. <u>Let $w \in \mathcal{M}(\mathcal{D})$ u be the exact solution of the problem $P(H_0,w,g)$ and U its finite element solution with respect to M. Then there exist strictly positive constants C_0, C_1 depending only on K, μ, λ and Ω such that</u>

$$(6.6) \qquad C_0 \sum_{P \in \mathcal{R}(\mathcal{D})} B(\eta_p,\eta_p) \leq |||e|||^2 \leq C_1 \sum_{P \in \mathcal{R}(\mathcal{D})} B(\eta_p,\eta_p)$$

<u>where</u>

$$(6.7) \qquad \eta_p \in \tilde{H}(\omega_p) = \{v \in H_0 | v=0 \text{ on } \Omega-\omega_p\}$$

<u>and</u>

$$(6.8) \qquad B(\eta_p,v) = B(e,v), \quad \forall v \in \tilde{H}(\omega_p) \qquad .$$

Proof. It follows from the definition of U and u that

$$B(e;v) = 0, \quad \forall v \in M_o$$

and therefore

$$B(e,e) = B(e,e-v) = B(e, \sum_{P \in \mathcal{R}(\mathcal{D})} v_p(e-v)) = \sum_{P \in \mathcal{R}(\mathcal{D})} B(\eta_p, v_p(e-v)) \quad .$$

Using Schwarz's inequality we get

$$B(e,e) \leq \sum_{P \in \mathcal{R}(\mathcal{D})} |||\eta_p||| \; |||v_p(e-v)||| \leq [\sum_{P \in \mathcal{R}(\mathcal{D})} |||\eta_p|||^2]^{1/2}$$

$$[\sum_{P \in \mathcal{R}(\mathcal{D})} |||v_p(e-v)|||^2]^{1/2} \quad .$$

By (6.2)

$$|||v_p(e-v)|||^2 \leq C||v_p(e-v)||^2_{H^1(\Omega)}$$

and therefore by Theorem 5.9 for a proper choice of v

$$\sum_{P \in \mathcal{R}(\mathcal{D})} |||v_p(e-v)|||^2 \leq C||e||^2_{H^1(\Omega)} \leq C|||e|||^2 \quad .$$

Hence we have

$$|||e|||^2 \leq C \sum_{P \in \mathcal{R}(\mathcal{D})} |||\eta_p|||^2$$

which proves right hand side of (6.6).

Let us prove now the left hand side of (6.6). Define

$$w_j = \sum_{P\in X_j} \eta_P \ , \qquad j = 1,\dots,J \ , \qquad \text{where}$$

X_j are the sets introduced in lemma 4.2. We have now

$$(6.9) \qquad B(e,w_j) = \sum_{P\in X_j} B(e,\eta_P) = \sum_{P\in X_j} B(\eta_P,\eta_P) \quad .$$

Because for $P \neq Q$, $P,Q \in X_j$ int (supp η_P) $\cap$ int (supp η_Q) $= \emptyset$ we have $B(\eta_P,\eta_Q) = 0$ and hence

$$(6.10) \qquad B(w_j,w_j) = \sum_{P\in X_j} B(\eta_P,\eta_P) \quad .$$

Further

$$|B(e,w_j)| \leq |||e|||\ |||w_j|||$$

and hence using (6.9) and (6.10) we get

$$|||w_j|||^2 = \sum_{P\in X_j} |||\eta_P|||^2 \leq |||e|||\ |||w_j|||$$

and hence

$$|||e|||^2 \geq |||w_j|||^2 = \sum_{P\in X_j} |||\eta_P|||^2 \quad .$$

Because j ranges over $1,\dots,J$ we get

$$(6.11) \qquad |||e|||^2 \geq \frac{1}{J} \sum_{P\in R(\mathcal{D})} |||\eta_P|||^2 \quad .$$

By lemma 4.2 $J \leq L(k)$ and therefore (6.11) proves the left side of (6.6).

We have proven in Theorem 6 that the estimator

$$\varepsilon^2(U) = \sum_{P\in R(\mathcal{D})} B(\eta_P,\eta_P)$$

is simultaneously on upper and lower estimators. The individual terms η_P are determined locally on separate stars ω_P . Let us underline that although the unknown error e is present in the definition of η_P in (6.8) we have not to know it. In fact

$$B(e,v) = B(U-u,v) = B(U,v) - B(u,v) = B(U,v) - (q,v) \quad .$$

Remark. The proof of the theorem 6.1 follows very closely the ideas in [9] .

We assumed in Theorem 6.1 that $w\in M(\mathcal{D})$. It is obvious that $w = (w_1,w_2)$ influences the solution only by its values at the boundary $\partial\Omega$, more exactly on Γ . In general when $w\notin M(\mathcal{D})$ we replace w by $\tilde{w}\in M(\mathcal{D})$ and estimate the norm of the solution of the problem $P(H_o,w-\tilde{w},0)$. Usually it is easy to explicitly construct a function $z\in H^1(\Omega) \times H^1(\Omega)$ $z_i = (w-\tilde{w})_i$ on Γ_i , $i = 1,2$ and the desired estimate is then simply $|||z|||$ In practical cases we can expect that $|||z|||$ is much smaller than $|||e|||$.

7. THE A-POSTERIORI ERROR ESTIMATE

Let us denote by Q any of the squares comprising the domain Ω and let Q_o be the unit square $[0,1] \times [0,1]$. Assume that $z_i \in H^1(Q_o)$, $i = 1,\dots,n$ are given and denote by Z the linear span of z_i, $i = 1,\dots n$.

Definition 7.1 Let $\rho > 0$, $\varepsilon > 0$. By $\P(Z,\rho,\varepsilon,Q)$ we denote the family of functions $\xi \in H^o(Q)$ such that for any square $S = [a_1,a_1+h] \times [a_2,a_2+h] \subset Q$ the following properties are fulfilled

α) $\xi \in H^1(S)$

β) There exists $\xi_o = z(h^{-1}(x-a))$, $z \in Z$ and constant M (both depending on ξ) such that

i) $||\xi-\xi_o||_{H^o(S)} \le M \cdot h$,

ii) $|\xi-\xi_o|_{H^1(S)} \le M$,

iii) $|\xi_o|_{H^1(S)} \ge \rho M h^{-\varepsilon}$.

Let us illustrate our definition by a few examples.

(i) Let Z be the set of all polynomials of degree less than or equal to n. Then any polynomial of degree $\le n$ on Q belongs to the family $\P(Z,\rho,\varepsilon,Q)$ with ρ and ε arbitrary.

ii) Let Z be the set of guadratic polynomials. Then $\xi = \sin x_1$ belongs to the family $\P(Z,\rho,\varepsilon,Q)$ for $\varepsilon \le 1$ and some suitably chosen ρ.

iii) The family $\mathfrak{F}$ is typically characterized by Z being the polynomials of degree $\leq m$, say, and then we take ξ_o to be a suitable Taylor expansion of ξ. Then (i) and (ii) are more or less standard, and (iii) states that ξ_o is not 'degenerate".

<u>LEMMA 7.1. Suppose that $f \in \pi(Z,\rho,\varepsilon,Q)$ then there exists C (dependent on (Z,ρ,ε,Q), such that for any square $S \subset Q$</u>

$$(7.1) \qquad \underline{|f|_{H^1(S)} \leq C(\operatorname{diam} S)^{-1} \inf_{\substack{d = \text{constant} \\ \text{functions on } S}} \|f-d\|_{H^0(S)}}$$

<u>Proof.</u> 1) Define $F \in H^1(Q_o)$ by

$$F(x) = f(a + hx)$$

Now by the assumption there exists $F_o(x) \in Z$ such that

$$(7.2) \qquad \|F-F_o\|_{H^o(Q_o)} \leq M \ ,$$

$$(7.3) \qquad |F-F_o|_{H^1(Q_o)} \leq M \ ,$$

$$(7.4) \qquad |F_o|_{H^1(Q_o)} \geq \rho M h^{-\varepsilon} \ .$$

Denote now by $\overline{F}$ respectively $\overline{F_o}$ the average of F respective to F_o on Q_o. Then we have

$$(7.5) \qquad \|(F-F_o)-(\overline{F-F_o})\|_{H^o(Q_o)} \leq \|F-F_o\|_{H^o(Q_o)} \leq M$$

because $\overline{F-F_o}$ is the H^o projection onto the set of constant functions on Q_o .

Because Z is finite dimensional space there exists C (dependent on Z) such that

$$(7.6) \qquad |F_o|_{H^1(Q_o)} = |F_o-\overline{F_o}|_{H^1(Q_o)} \le C||F_o-\overline{F_o}||_{H^o(Q_o)}$$

and hence combining (7.4), (7.5) and (7.6) we get

$$M \le h^{\varepsilon}\rho^{-1}|F_o|_{H^1(Q_o)} \le Ch^{\varepsilon}\rho^{-1}||F_o-\overline{F_o}||_{H^o(Q_o)}$$

and

$$||(F-F_o)-(\overline{F-F_o})||_{H^o(Q_o)} \le C\rho^{-1}h^{\varepsilon}||F_o-\overline{F_o}||_{H^o(Q_o)} \quad .$$

On the other hand

$$||F-\overline{F}||_{H^o(Q_o)} \ge ||F_o-\overline{F_o}||_{H^o(Q_o)} - ||(F-F_o)-(\overline{F-F_o})||_{H^o(Q_o)}$$

and hence for $C\rho^{-1}h^{\varepsilon} < 1/2$ we have

$$(7.7) \qquad ||F-\overline{F}||_{H^o(Q_o)} \ge \frac{1}{2}||F_o-\overline{F_o}||_{H^o(Q_o)} \quad .$$

Further from (7.3) and (7.4)

(7.8) $$|F|_{H^1(Q_o)} \le |F_o|_{H^1(Q_o)} + |F_o-F|_{H^1(Q_o)} < |F_o|_{H^1(Q_o)}$$

$$+ M \le |F_o|_{H^1(Q_o)}(1 + \rho^{-1}h^{\varepsilon}) \le 2|F_o|_{H^1(Q_o)}$$

for $h^{\varepsilon}\rho^{-1} < 1$.

Therefore by (7.6), (7.7) and (7.8), we have

$$|F|_{H^1(Q_o)} \le 2|F_o|_{H^1(Q_o)} \le 2C||F_o-\overline{F}_o||_{H^o(Q_o)} \le 4C||F-\overline{F}||_{H^o(Q_o)}$$

so upon rescaling back to S (7.1) is proven in the case $\text{diam } S \le h_o(\rho,\varepsilon,Z)$.

2) Suppose now that $\text{diam } S > h_o(\rho,\varepsilon,Z)$. Put

$$\sigma = \left[\frac{\text{diam } S}{h_o}\right]^{INT} + 1$$

where $[\cdot]^{INT}$ denotes as before the integral part. Clearly we can divide S into σ^2 congruent squares S_i, $i = 1,\dots,\sigma^2$ with

$$\text{diam } S_i = \frac{\text{diam } S}{\sigma} \le h_o \quad .$$

Thus

$$|f|^2_{H^1(S)} = \sum_{i=1}^{\sigma^2} |f|^2_{H^1(S_o)} \le C\left(\frac{\text{diam } S}{\sigma}\right)^{-2} \sum_{\substack{i=1 \\ d=\text{constant} \\ \text{function}}}^{\sigma^2} \inf||f-d||^2_{H^o(S_i)}$$

$$\le C\sigma^2(\text{diam } S)^{-2} \inf_{\substack{d=\text{constant} \\ \text{function}}} ||f-d||^2_{H^o(S)}$$

But $\sigma \leq 1 + \frac{\text{diam } Q}{h_o} \leq C$ and the result follows.

We introduce the family $\P(\mathcal{Z},\rho,\varepsilon,Q)$ for $Q = Q_\theta^k$ with Q_θ^k as in section 2. Because Ω consists of a finite number of Q_θ^k it is clear that we can extend the family $\P$ into $\P(\mathcal{Z},\rho,\varepsilon,\Omega)$ so that the restriction on $Q = Q_\theta^k$ is the family $\P(\mathcal{Z},\rho,\varepsilon,\Omega)$.

In what follows we will assume that we are concerned with problems $P(H_o,w,g)$ introduced in section 6 where $g \equiv (g_1,g_2)$, $g_i \in \P(\mathcal{Z},\rho,\varepsilon,\Omega)$.

We shall discuss now the error estimate of section 6 in more detail. We have shown in theorem 6.1 that the essential part of the estimate is the norm of η_p which is defined on the star ω_p . In section 4 we introduced the standard star Φ_p (see lemma 4.3) as the image of ω_p under the mapping J_p .

On Φ_p we will define now the space $\hat{H}_p = \hat{H}_{1,p} \times \hat{H}_{2,p}$ with

$$\hat{H}_p(\Phi_p) = \{v = (v_1,v_2) \in H^1(\Phi_p) \times H^1(\Phi_p) | v \circ J_p \in \tilde{H}(\omega_p)\}$$

with $\tilde{H}(\omega_p)$ defined in (6.7).

In 6.1 we defined the bilinear form B . Let us define the form B_p defined on $\hat{H}_p \times \hat{H}_p$ with the same expression as in (6.1) but with integration over Φ_p , and $\hat{\lambda}, \hat{\mu}$ instead of λ, μ, $\hat{\lambda} = \lambda \circ J^{-1}$, $\hat{\mu} = \mu \circ J^{-1}$.

<u>LEMMA 7.2.</u> <u>The bilinear form B_p is such that for any $u \in \hat{H}_p$,</u>

$$(7.9) \qquad \underline{C_1 ||u||^2_{H^1(\Phi_p)} \leq B_p(u,u) \leq C_2 ||u||^2_{H^1(\Phi_p)}}$$

<u>with constants C_i, $i = 1,2$ dependent only on λ,μ and K but independent of P</u> .

Proof. For each $P \in \mathcal{R}(\mathcal{D})$ there are line segments $\Gamma^{(i)} \subset \partial\Phi_p$ $(i = 1,2)$ both being edges of standard P elements of Φ_p and such that $u \in \hat{H}_p \Rightarrow u_i = 0$ on $\Gamma^{(i)}$. Applying now lemma 4.3 we see that there are only finite number of different cases of the domains Φ_p and the line segments $\Gamma^{(i)}$ for each the Korn inequality holds. Therefore (7.9) holds with $|\cdot|_{H^1(\Phi_p)}$ replacing $||\cdot||_{H^1(\Phi_p)}$. Because these two norms are equivalent with constants depending on $(\Phi_p, \Gamma^{(1)}, \Gamma^{(2)})$ we get (7.9) immediately when using again lemma 4.3.

Denote now by $\hat{M}_p(\Phi_p)$ the set of all $u \in \hat{H}_p(\Phi_p)$ being bilinear on every standard P element of the standard star Φ_p. Further we denote

$$\hat{e} = e \circ J_p^{-1}, \quad \hat{u} = u \circ J_p^{-1}, \quad \hat{U} = U \circ J_p^{-1}, \quad \hat{g}_i = g_i \circ J_p^{-1}, \quad \hat{\hat{g}}_i = \hat{g}_i(\text{diam}\,\Delta_p)^2$$

$$i = 1,2$$

where e, u, U, g were defined in section 6.

We have

$$B_p(\hat{e}, v) = 0, \quad \forall v \in \hat{M}_p(\Phi_p)$$

and

$$B_p(\hat{e}, v) = B_p(\hat{U}, v) - B_p(\hat{u}, v)$$

with

$$B_p(\hat{u}, v) = \sum_{i=1}^{2} (\hat{\hat{g}}_i, v)_{L_2(\Phi_p)}, \quad \forall v \in \hat{H}_p$$

By integration by parts we get

$$B_p(\hat{U},v) = \sum_{\Delta\in S_p} \sum_{i=1}^{2} \left[\sum_{j=1}^{2} \int_{\partial\Delta} \sigma_{i,j}(U) n_i v_j ds - \int_{\Delta} L_i(\hat{U}) v_i dx \right]$$

where (n_1,n_2) is the unit normal on each edge of $\partial\Delta$ pointing outward from Δ and

$$\sigma_{i,j} = \hat{\lambda}(\frac{\partial \hat{U}_1}{\partial x_1} + \frac{\partial \hat{U}_2}{\partial x_2}) + \mu\left[\frac{\partial \hat{U}_i}{\partial x_j} + \frac{\partial \hat{U}_j}{\partial x_i}\right]$$

$$\hat{L}_1(\hat{U}) = (\hat{\lambda}+\hat{\mu}) \frac{\partial^2 \hat{U}_2}{\partial x_1 \partial x_2}$$

$$\hat{L}_2(\hat{U}) = (\hat{\lambda}+\hat{\mu}) \frac{\partial^2 \hat{U}_1}{\partial x_1 \partial x_2}$$

Let $\Gamma_{1,p}$ (respective $\Gamma_{2,p}$) be the union of all vertical (horizontal) edges of all $\Delta \in S_p$. Then

$$(7.10) \qquad B_p(\hat{e},v) = \sum_{i=1}^{2} \left[-([\hat{\hat{g}}_i + \hat{L}_i(\hat{U})], v_i)_{L_2(\Phi_p)} + \sum_{j=1}^{2} (\hat{J}_i(\sigma_{i,j}(\hat{U})), v_j)_{L_2(\Gamma_{i,p})} \right]$$

where J_i indicates the jump in a function across Γ_i . The jump J_i has an obvious sense if the relevant edge Γ of Δ is inside Φ_p . If $\Gamma \subset \partial\Phi_p$ we have to distinguish the case of whether $v_j \in H_{1,p} \Rightarrow v_j = 0$ on Γ or not. If all $v_j = 0$ on Γ we will take the jump to be zero. If it is not the case, then σ_{ij} is set equal to zero outside Φ_p , and the jump is taken the usual sense. This convention will be used through the paper.

Because $\hat{U}$ is bilinear on every $\Delta \in \Phi_p$ and $\hat{\lambda},\hat{\mu}$ are constant on Δ we see that $L_i(\hat{U})$ are constant on every Δ and $J_i(\sigma_{i,j}(\hat{U}))$ are linear on every edge of Δ .

Putting $\hat{\eta}_p = \eta_p \circ J_p^{-1}$ (6.8) gives

$$B_p(\hat{\eta}_p, v) = B_p(\hat{e}, v) \quad \forall v \in \hat{H}_p(\Phi_p) \quad . \tag{7.11}$$

Using Lemma 7.2 and proceeding in the standard fashion it is easily seen that $\hat{\eta}_p$ is the unique $\hat{H}_p(\Phi_p)$ function satisfying (7.11).

LEMMA 7.3. *There exists a constant C dependent only on K and μ, λ such that*

$$\|\eta_p\|_{H^1(\Phi_p)} \leq C\left[\sum_{i=1}^{2} (\|\hat{\hat{g}}_i + \hat{L}_i(\hat{U}))\|_{H^o(\Phi_p)} + \sum_{j=1}^{2} \|\hat{J}_i(\sigma_{ij}(\hat{U})\|_{H^o(\Gamma_i)})\right] . \tag{7.12}$$

Proof. The set of possible S_p's is finite. Therefore we have by Sobolev imbedding theorem

$$\|v\|_{H^o(\Gamma_i)} \leq C\|v\|_{H^1(\Phi_p)} , \tag{7.13}$$

with C depending only on K . (7.13) with the Lemma 7.2 and Schwarz's Inequality leads to the desired result.

LEMMA 7.4. *Assume that $g_i \in \P(Z, \rho, \varepsilon, \Omega)$ then*

$$\|\hat{\eta}_p\|_{H^1(\Phi_p)} \geq C \sum_{i=1}^{\ell} (\|\hat{\hat{g}}_i + \hat{L}_i(\hat{U})\|_{H^o(\Phi_p)} + \sum_{j=1}^{2} \|\hat{J}_i(\sigma_{ij}(\hat{U}))\|_{H^o(\Gamma_i)}) \tag{7.14}$$

where C depends only on K , the family ¶ and λ, μ .

Proof. Suppose that the constant C does not exist. Then there exist for each $n = 1,2,3,\ldots$

i) a K-mesh $\mathcal{D}^{[n]}$

ii) $p^{[n]} \in R(\mathcal{D}^{[n]})$

iii) $\hat{U}^{[n]} \in H^1(\Phi_{p[n]})$ with $\hat{U}^{[n]}$ being bilinear on each $\Delta \in S_{p[n]}$

iv) $\hat{g}_i^{[n]} \in L_2(\Phi_{p[n]})$ with $|\hat{g}_i^{[n]}|_{H^1(\Delta)} \le C\|\hat{g}_i^{[n]} - {}^{\Delta}\bar{\hat{g}}^{[n]}\|_{H^o(\Delta)}$ for any $\Delta \in S_{p[n]}$, with C independent of n and ${}^{\Delta}\bar{\hat{g}}$ denotes the average of $\hat{g}_i^{[n]}$ on Δ

such that the unique solution $\eta^{[n]} \in \hat{H}_{p[n]}(\Phi_{p[n]})$ of

$$(7.15)\qquad B_{p[n]}(\hat{\eta}^{[n]},v) = \sum_{i=1}^{2} [-(\hat{g}_i^{[n]} + \hat{L}_i(\hat{U}^{[n]}),v_i)_{L_2(\Phi_{p[n]})}$$

$$+ \sum_{j=1}^{2} (\hat{J}_i(\sigma_{i,j}(\hat{U}^{[n]})),v_j)_{L_2(\Gamma_i^{[n]})}] \quad \forall v \in H_{p[n]}(\Phi_{p[n]}) ,$$

satisfies

$$(7.16)\qquad \|\hat{\eta}^{[n]}\|_{H^1(\Phi_{p[n]})} \le \frac{1}{n} \sum_{i=1}^{2} (\|\hat{g}_i^{[n]} + \hat{L}_i(\hat{U}^{[n]})\|_{H^o(\Phi_{p[n]})}$$

$$+ \sum_{j=1}^{2} \|\hat{J}_i(\sigma_{i,j}(\hat{U}^{[n]}))\|)_{H^o(\Gamma_{i_1}^{[n]})} .$$

Putting $r_i^{[n]} = \hat{g}_i^{[n]} + \hat{L}_i(\hat{U}^{[n]})$

$$\xi_{i,j}^{[n]} = \hat{J}_i(\sigma_{i,j}(\hat{U}^{[n]}))$$

then without loss of generality we can assume

$$(7.17) \qquad \sum_{i=1}^{2} (||r_i^{[n]}||_{H^o(\Phi_{p[n]})} + \sum_{j=1}^{2} ||\xi_{i,j}^{[n]}||_{H^o(\Gamma_i^{[n]})}) = 1$$

and

$$(7.18) \qquad |r_i^{[n]}|_{H^1(\Delta)} \leq C||r_i^{\Delta} - \overset{\Delta}{\bar{r}}{}_i^{[n]}||_{H^o(\Delta)} \qquad \forall \Delta \in S_{p^{[n]}} \quad .$$

The inequality (7.18) follows from the assumption that $g_i \in \P(Z,\rho,\varepsilon,Q)$ and that $\hat{L}_i(\hat{U}^{[n]})$ is constant on Δ . From (7.17) we have

$$(7.19) \qquad ||r_i^{[n]} - \overset{\Delta}{\bar{r}}{}_i^{[n]}||_{H^o(\Delta)} \leq ||r_i^{[n]}||_{H^o(\Delta)} \leq ||r_i^{[n]}||_{H^o(\Phi_p)} \leq 1 \quad .$$

Using Lemma 4.3 we may assume that $\Phi_{p(n)} = \Phi$, $S_{p^{[n]}} = S$, $\hat{H}_{p^{[n]}}(\Phi_{p^{[n]}}) = \hat{H}$. Further by Rellich's lemma,(7.18) and (7.19), we may also assume that $r_i^{[n]} \to \tilde{r}_i$ in $H^o(\Phi)$.

Now $\xi_{i,j}^{[n]}$ is linear on each edge of $\Delta \in S$ and so we may assume also that $\xi_{i,j}^{[n]} \to \tilde{\xi}_{i_j}$ in $H^o(\hat{\Gamma})$ where $\hat{\Gamma}$ is the union of all edges of $\Delta \in S$. (7.17) yields

$$(7.20) \qquad \sum_{i=1}^{2} (||\tilde{r}_i||_{H^o(\Phi)} + \sum_{j=1}^{2} ||\tilde{\xi}_{i,j}||_{H^o(\hat{\Gamma})}) = 1 \quad .$$

(7.15), (7.16) and (7.20) leads now to the contradiction.

<u>LEMMA 7.5. Let $p \in \mathcal{R}(\mathcal{D})$ be such that $\bar{v}_p \equiv (v_p, v_p) \in \tilde{H}(\omega_p)$ (see(6.7)) and $\omega_p \subset Q_\theta^k$ for some $k \in Z_o$. Then</u>

$$\|\hat{\eta}_p\|_{H^1(\Phi_p)} \leq C \sum_{i=1}^{2} \left(\|\hat{\hat{g}}_i - {}^{\Delta}\overline{\hat{\hat{g}}}_i\|_{H^o(\Phi_p)} + \sum_{j=1}^{2} \|\hat{J}_i \sigma_{i,j}(\hat{U})\|_{H^o(\Gamma_i)}\right)$$

<u>where ${}^{\Delta}\overline{\hat{\hat{g}}}_i$ restricted to any $\Delta \subset \Phi_p$ is the average value of $\hat{\hat{g}}_i$ on Δ.</u>

<u>Proof.</u> The proof will be by contradiction. Suppose the lemma does not hold. Then for each $n = 1,2,3,\ldots$ we can find $\mathcal{D}^{[n]}$, $p^{[n]}$, $U^{[n]}$ and $\hat{g}_i^{[n]}$ as in i), iii), and iv) of the proof of lemma 7.4 when ii) is replaced by ii')

ii') $\quad p^{[n]} \in \mathcal{R}(\mathcal{D}^{n})$, $(v_{p^{[n]}}, v_{p^{[n]}}) \in \tilde{H}(\omega_{p^{[n]}})$, $\omega_{p^{[n]}} \subset Q_\theta^k$ for some $k \in Z$

and (because of (7.10) and $B_p(\hat{e},v) = 0$)

$$(7.21) \qquad -(\hat{\hat{g}}^{[n]} + \hat{L}_i(\hat{U}^{[n]}), v_{p^{[n]}} \circ J^{-1}_{p^{[n]}})_{H^o(\Phi_{p^{[n]}})}$$

$$+ \sum_{j=1}^{\ell} (\hat{J}_i(\sigma_{i,j}(\hat{U}^{[n]})), v_{p^{[n]}} \circ J^{-1}_{p^{[n]}})_{H^o(\Gamma_{i,p^{[n]}})} = 0$$

and with $\eta^{[n]}$ as in (7.15) and

$$(7.22) \qquad \|\eta^{[n]}\|_{H^1(\Phi_{p[n]})} \geq n \Bigg[\sum_{i=1}^{n} \left(\|\hat{\hat{g}}_i^{[n]} - {}^{\Delta}\overline{\hat{\hat{g}}}{}^{[n]}\|_{H^o(\Phi_{p[n]})}\right.$$

$$\left. + \sum_{j=1}^{2} \|\hat{J}_i(\sigma_{i,j}(\hat{U}^{[n]}))\|_{H^o(\Gamma_{i,p^{[n]}})}\right) \Bigg]$$

As in the proof of the previous theorem, by use of lemma 4.3 we can assume that

α) $S_{p[n]} = S$, $\Phi_{p[n]} = \Phi$, $\Gamma_{i,p[n]} = \Gamma_i$, $\hat{H}_{p[n]} = \hat{H}$ are independent of n

β) $\omega_{p[n]} \subset Q_\theta^k$ for some $k \in Z_o$ where k is independent of n (because Z_o is finite).

Now α), β) yield that also $B_{p[n]}(\cdot,\cdot) = \hat{B}(\cdot,\cdot)$ does not depend on n

γ) $||\hat{\eta}^{[n]}||_{H^1(\Phi)} = 1$.

Let us set now

$$r_i^{[n]} = \hat{g}_i^{[n]} + \hat{L}_i(\hat{U}^{[n]})$$

$$\xi_{i,j} = \hat{J}_i(\sigma_{i,j}(\hat{U}^{[n]})) \quad .$$

Now by (7.21) and γ we have

$$(7.23) \quad \begin{aligned} &||r_i^{[n]} - {}^{\Delta}\overline{r}_i^{[n]}||_{H^o(\Delta)} \to 0 \ , \qquad \forall \Delta \in \Phi \\ &||\xi_{i,j}||_{H^o(\Gamma_i)} \to 0 \quad . \end{aligned}$$

It is easily demonstrated that

$$(7.24) \quad \sum_{i=1}^{2} ||\hat{L}_i(\hat{U}^{[n]}) - {}^{\varphi}\overline{\hat{L}_i(\hat{U}^{[n]})}||_{H^o(\Phi)} \leq \sum_{i,j=1}^{2} ||\xi_{i,j}||_{H^o(\Gamma_i)}$$

and so we must have

$$(7.25) \qquad \| \hat{L}_i(U^{[n]}) - \overline{{}^{\Phi}L_i(U^{[n]})} \|_{H^o(\Phi)} \to 0 \quad .$$

By (7.23), using lemma 7.1 Rellich's lemma and the fact that S is finite, we may assume that

$$(7.26) \qquad r_i^{[n]} \to \tilde{r}_i \quad \text{in} \quad H^o(\Phi)$$

$$(7.27) \qquad \hat{g}_i^{[n]} \to \tilde{g}_i \quad \text{in} \quad H^o(\Phi) \quad .$$

Because of lemma 7.1 we see that $\hat{g}_i^{[n]} \to \tilde{g}_i$ also in $H^{3/4}(\Phi)$ where $H^{3/4}(\Phi)$ is the usual space of fractional derivatives. Now by (7.23) again we see that $\tilde{r}_i$ and $\tilde{g}_i$ must be constant on each $\Lambda \in S$. Further by lemma 4.3(i) and the fact that $\tilde{g}_i \in H^{3/4}(\Phi)$ we see that $\tilde{g}_i$ is in fact constant over all of Φ .

By (7.26) and (7.27), it follows that $\hat{L}_i(\hat{U}^{[n]})$ converges in $H^o(\Phi)$ and by (7.25) it must converge to a constant on Φ . This follows that $\tilde{r}_i$ is a constant on Φ .

By (7.21) in the limit we see that

$$(\tilde{r}_i, v_{p[n]} \circ J^{-1}_{p[n]})_{H^o(\Phi)} = 0 \quad .$$

By lemma 3.5 we have $v_{p[n]} \geq 0$ and $v_{p[n]} > 0$ on a set of positive measure. Thus we conclude that $\tilde{r}_i = 0$.

Returning to 7.15, we see that $\eta^{\{n\}} \to 0$ in $H^1(\Phi)$ which contradicts γ .

Now we return to the question of the a-posteriori estimate. Given a mesh $\mathcal{D}$ and the corresponding finite element solution U as in (6.5), we will associate to every $\Delta \in \mathcal{D}$ the error indicator $\bar{\eta}(\Delta)$ defined by

$$(7.28) \qquad \bar{\eta}^2(\Delta) = \sum_{i=1}^{2} \{\mathrm{diam}\, \Delta \sum_{j=1}^{2} \| J_i \sigma_{ij}(U) \|^2_{H^0(\partial_i \Delta)} + (\mathrm{diam}\, \Delta)^2 \| g_i + L_i(U) \|^2_{H^0(\Delta)} \} \quad .$$

where

$$L_1(U) = (\lambda+\mu) \frac{\partial^2 U_2}{\partial x_1 \partial x_2}$$

$$L_2(U) = (\lambda+\mu) \frac{\partial U_1}{\partial x_1 \partial x_2}$$

$$\sigma_{i,j} = \lambda\left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2}\right) + \mu\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right]$$

and J_1 (J_2) indicates the jump across the vertical edges, $\partial_1 \Delta$ (horizontal edges, $\partial_2 \Delta$) of Δ . We use the same convention as before, i.e. if the edge $\Gamma \subset \partial\Omega$ is such that $v_i \in H_i \to v_i = 0$ on Γ then the jump J_i is not taken into consideration. If it is not the case, we take the jump $J_i(\sigma_{i,j}) = \sigma_{i,j}$.

The estimator $\bar{E}$ then is defined

$$\bar{E}^2 = \sum_{\Delta \in \mathcal{D}} \bar{\eta}^2(\Delta) \qquad .$$

<u>THEOREM 7.6</u>. <u>The estimator $\bar{E}$ is an upper and lower estimator, i.e. there exists c_1, c_2 dependent only on K, λ, μ, Ω and ¶ but not u and $\mathcal{D}$, such</u>

that the error of the finite element solution of the problem $P(H_o,w,g)$ $w \in [M(\mathcal{D})]^2$, $g \in \P(\mathcal{Z},\rho,\varepsilon,\Omega)$ satisfies the estimate

$$\bar{c}_1\bar{E} \leq ||e||_{H^1(\Omega)} \leq c_2\bar{E} \quad . \tag{7.30}$$

Before proving the theorem let us prove two simple lemma.

LEMMA 7.7. Suppose Γ is an edge of some $\Delta \in \mathcal{D}$. Then

α) Either (i) $\Gamma \subset \partial\Omega$,

or (ii) $\exists\, P \in R(\mathcal{D})$

such that $\Delta \subset \omega_P$ but $\Gamma \not\subset \partial\omega_P$.

β) $\Gamma \cap \Delta' \neq 0$ for at most χ elements $\Delta' \in \mathcal{D}$ where χ depends only on K .

Proof.

α) By lemma 3.5 we have $\Sigma v_p = 1$, so certainly for some $P \in R(\mathcal{D})$ $v_p > 0$ at the midpoint of Γ . Since v_p is a non negative and linear on Γ, then $v_p > 0$ on Γ except perhaps at an endpoint. The result is then immediate upon noting that v_p is continuous and inducing lemma 3.6.

β) The number of such elements is clearly bounded by the number with non empty intersection with Δ . Lemma 4.1 iii) then gives the result

LEMMA 7.8. Suppose $a_1,\ldots,a_m$ are non negative real numbers and let α be a mapping from $\{1,\ldots,n\} \to \{1,\ldots,m\}$ such that

i) α is onto

ii) $\exists M > 0$ such that $\forall 1 \leq j \leq m$ the set $\{i \mid 1 \leq i \leq n, \alpha(i) = j\}$ has at most M elements then

$$\sum_{j=1}^{m} a_j \leq \sum_{i=1}^{n} a_{\alpha(i)} \leq M \sum_{j=1}^{m} a_j \quad .$$

The lemma is obvious.

Proof of the Theorem 7.6. Using a scaling argument applied to lemma 7.3 and 7.4 we obtain

$$(7.31) \qquad \|\eta_p\|^2_{H^1(\omega_p)} \sim \sum_{\substack{\Delta\in\mathcal{D}\\ \Delta\subset\omega_p}} \sum_{i=1}^{2} [(\mathrm{diam})^2 \|g_i + L_i(U)\|^2_{H^0(\Delta)} +$$

$$+ \mathrm{diam}\,\Delta \sum_{j=1}^{2} \|J_i^o(\sigma_{ij}(U))\|^2_{H^0(\partial_i \Delta)}]$$

where J_i^o indicates that we use the jump across edges Δ which are on $\partial\omega_p$ but within Ω equal to zero. Adding over all $P \in R(\mathcal{D})$ and using lemma 7.8 and lemma 4.1, we are in a situation covered by lemma 7.7 and the result follows.

Before formulating the next theorem, we prove

LEMMA 7.10. Let $R^*(\mathcal{D}) = \{P \in R(\mathcal{D}) \mid v_p = 0 \text{ on } \bigcup_{k\in Z_o} \partial Q_\theta^k\}$. Then

$$(7.32) \qquad \bigcup_{P\in R^*(\mathcal{D})} \omega_p = \bar{\Omega} \quad .$$

Proof. The lemma follows from lemma 3.2 applied to each Q_θ^k separately which shows that any $\Delta \in \mathcal{D}$ has at least one vertex P say, being a regular node located in the interior of a Q_θ^k. Clearly the corresponding $\omega_p \subset Q_\theta^k$, and so $P \in R^*(\mathcal{D})$.

Let us introduce now another error indicator $\tilde{\eta}(\Delta)$,

$$(7.33) \qquad \tilde{\eta}^2(\Delta) = \sum_{\ell=1}^{2} \{\text{diam } \Delta \sum_{j=1}^{2} ||J_i\sigma_{i,j}(U)||^2_{H^o(\partial_i\Delta)} + (\text{diam } \Delta)^2 ||g_i - {}^{\Delta}\bar{g}_i||^2_{H^o(\Delta)}\}$$

where $\bar{g}_i$ is the average value of g_i on and the corresponding error estimator

$$(7.34) \qquad \tilde{E}^2 = \sum_{\Delta \in \mathcal{D}} \tilde{\eta}_i^2(\Delta) \quad .$$

THEOREM 7.11. The estimator (7.34) is also an upper and lower estimator.

Proof. Let $P \in R^*(\mathcal{D})$. Then using lemma 7.5 and 7.4, we see that for any $\Delta \in \omega_p$

$$(7.35) \qquad \sum_{i=1}^{2} (\text{diam } \Delta)^2 ||g_i + L_i(U)||^2_{H^o(\Delta)} \leq C \sum_{\Delta \in \omega_p} \sum_{i=1}^{2} ((\text{diam } \Delta)^2 ||g_i - {}^{\Delta}\bar{g}_i||_{H^o(\Delta)} +$$

$$+ \text{diam } \Delta \sum_{j=1}^{2} ||J_i^o\sigma_{i,j}(U)||^2_{H^o(\partial_i\Delta)}) \quad .$$

We have shown in lemma 4.1 that any Δ is contained in not more than $M(K)$ stars ω_p. Lemma 7.10 shows that every Δ is contained at least in one ω_p

$P \in R^*(\mathcal{D})$ and therefore summing (7.35) over all $\Delta \in \mathcal{D}$ we see immediately that $\bar{E} \le C\tilde{E}$ which yields that $\tilde{E}$ is an upper estimator. Because $\tilde{E} < \bar{E}$ $\tilde{E}$ is clearly a lower estimator.

Lemma 7.12. There exist $C_i > 0$, $i = 1,2$ such that

$$\text{(7.36)} \qquad C_1 \| J_i \sigma_{i,j}(U) \|^2_{H^0(\partial_i \Delta)} \le \operatorname{diam} \Delta \sum_{\ell=1}^{4} [J_i \sigma_{i,j}(U)(x_\ell^{(\Delta)})]^2 \le C_2 \| J_i \sigma_{i,j}(U) \|^2_{H^0(\partial_i \Delta)}$$

where $x_\ell^{(\Delta)}$ are the vertices of Δ.

Proof. The inequality (7.36) follows immediately from the fact that $J_i \sigma_{i,j}(U)$ is linear on every edge of Δ.

Lemma 7.12 allows us to introduce another error indicator,

$$\text{(7.37)} \qquad \eta^2(\Delta) = \sum_{i=1}^{2} \left\{ (\operatorname{diam} \Delta)^2 \sum_{j=1}^{2} \sum_{\ell=1}^{4} [J_i \sigma_{i,j}(U)(x_\ell^\Delta)]^2 + (\operatorname{diam} \Delta)^2 \| g_i - {}^\Delta\bar{g}_i \|^2_{H^0(\Delta)} \right\}$$

and error estimator

$$\text{(7.38)} \qquad E^2 = \sum_{\Delta \in \mathcal{D}} \eta_i^2(\Delta)$$

and we have from lemma 7.12 and Theorem 7.11,

THEOREM 7.13. The estimator (7.38) is an upper and lower estimator.

8. THE ASYMPTOTIC ESTIMATE

We have shown in the previous section that the estimator $\mathcal{E}$ introduced in (7.35) and (7.34) is under the proper assumptions simultaneously an upper and lower estimator.

In this section we will analyze an estimator E , which will be equivalent with $\mathcal{E}$ i.e. there exist C_1 and C_2 so that

$$(8.1) \qquad C_2\mathcal{E} \leq E \leq C_1\mathcal{E} \quad ,$$

and will be asymptotically exact for the energy norm $|||\cdot|||$ introduced in (6.3). We shall say that an estimator E is asymptotically exact with respect to the energy norm if

$$(8.2) \qquad \frac{E}{|||e|||} \to 1 \quad \text{as} \quad ||e|| \to 0 \qquad .$$

To show (8.2) we have to make various assumptions about the solution u and the meshes in addition to the assumption that the mesh is a K-mesh.

Suppose $\mathcal{D}(\Omega)$ is a mesh and let $\mathcal{D}' \subset \mathcal{D}$ satisfy

i) If $\Delta \in \mathcal{D}'$, diam $\Delta = h$, $h > 0$

ii) if $\Delta \in \mathcal{D}'$ then all vertices of Δ are proper nodes.

If (i), and (ii) hold then we shall say that $\mathcal{D}'$ is uniform. If only (i) holds then we will say that $\mathcal{D}'$ has uniform size.

Suppose now that $A \subset B \subset \Omega$ then we shall write $A < B$ if

$$P(A,B) = \operatorname{dist}(A, \mathbb{R}^2 - B) > 0 \ .$$

Let $\mathcal{D}_o \subset \mathcal{D}(\Omega)$ then we will write

$$\Omega(\mathcal{D}_0) = \mathrm{Int} \bigcup_{e\in\mathcal{D}_0} \bar{e}$$

and $N(\mathcal{D}_0)$ = number of elements contained in $\mathcal{D}_0$. We will write $\mathcal{D}'' < \mathcal{D}'$ if $\Omega(\mathcal{D}'') \subset \Omega(\mathcal{D}')$.

Finally denote by $M(\mathcal{D};\Omega_1)$ the set of functions of $[M(\mathcal{D})]^2$ restricted to Ω_1 and $\overset{\circ}{M}(\mathcal{D},\Omega_1) \subset M(\mathcal{D},\Omega_1)$ the set of functions which are zero at $\partial\Omega_1$.

We shall make use of the following version of Theorem 5.2 of [11].

THEOREM 8.1. Let $\mathcal{D}' \subset \mathcal{D}(\Omega)$ be a uniform mesh. Assume that the bilinear form B defined in (6.1) has constant coefficients on $\Omega(\mathcal{D}')$ and let $\mathcal{D}'' < \mathcal{D}'$. Then if $\xi = (\xi_1, \xi_2) \in [H^1(\Omega(\mathcal{D}'))]^2$ satisfies

$$|B(\xi,\phi)| \le A \sum_{i=1}^{2} |\phi_i|_{H^1(\Omega(\mathcal{D}'))} \tag{8.3}$$

for all $\phi_i \in \overset{\circ}{M}(\mathcal{D},\Omega(\mathcal{D}'))$ then

$$|\xi_i|_{H^1(\Omega(\mathcal{D}''))} \le C \sum_{j=1}^{2} \left\{ \inf_{\chi\in M(\mathcal{D},\Omega(\mathcal{D}))} |\xi_j - \chi|_{H^1(\Omega(\mathcal{D}'))} + \frac{1}{\rho(\Omega(\mathcal{D}''),\Omega(\mathcal{D}'))} |\xi_j - \chi|_{H^0(\Omega(\mathcal{D}'))} + \right.$$
$$\left. + \frac{1}{\rho(\Omega(\mathcal{D}''),\Omega(\mathcal{D}'))} |\xi_j|_{H^0(\Omega(\mathcal{D}'))} + A \right\} \tag{8.4}$$

with C dependent only on the bilinear form (6.1) (and not on $\Omega(\mathcal{D}'')$, $\Omega(\mathcal{D}')$).

Suppose now that $\mathcal{D}' \subset \mathcal{D}$ has uniform mesh size and let $\mathcal{D}'' < \mathcal{D}'$. Then we can define a difference operator T_i^h which maps $H^1(\Omega(\mathcal{D}'))$ into $H^1(\Omega(\mathcal{D}''))$ so that

$$(T_i^h u)(x) = 2h^{-1}[u(x+he_i) - u(x-he_i)]$$

with h being the diameter of the elements of $\mathcal{Q}'$ and e_i the coordinate unit vector.

We have then the following theorem which follows easily from Theorem 6.2 of [ ] .

THEOREM 8.2. Suppose that u_j and U_j $j = 1,2$ are the components of the exact respective finite element solution of the problem $P(H_o,w,g)$ as defined in Section 6.

Assume further that $\mathcal{D}'' < \mathcal{D}'$, $\mathcal{D}' \subset \mathcal{D}$ are as above, that the bilinear form B of (6.1) has constant coefficients on $\Omega(\mathcal{D}')$ and $u_j \in H^3(\Omega(\mathcal{D}'))$, $j = 1,2$. Then if $\mathcal{D}''' < \mathcal{D}''$ it follows that for $D_i = \frac{\partial}{\partial x_i}$ we have

$$(8.5)\qquad ||D_i u_j - T_i^h U_j||_{H^o(\Omega(\mathcal{D}'''))} \le C \sum_{k=1}^{2} (h^2 ||u_k||_{H^3(\Omega(\mathcal{D}'))} + \frac{1}{\rho(\Omega(\mathcal{D}'''),\Omega(\mathcal{D}''))} ||u_k - U_k||_{H^o(\Omega(\mathcal{D}'))}) .$$

Denote now for any square $Q \subset \mathbb{R}^2$ by $H_i(Q)$ the space of all functions which are of the form αx_i on Q (when referred to an origin placed at the center of Λ) .

Now we prove

THEOREM 8.3. Using the same notation as in Theorem 8.2 then for each $\Lambda \in \mathcal{D}'''$ there exists $\phi_{j,i}^{\Delta} \in H_i(\Delta)$ $i,j = 1,2$ such that

$$(8.6)\qquad ||D_i(u_j - U_j) - \sum_{\Lambda \in \mathcal{D}'''} \phi_{j,i}||_{H^o(\Omega(\mathcal{D}'''))} \le C\rho^{-1}(\Omega(\mathcal{D}'''),\Omega(\mathcal{D}'')) \sum_{k=1}^{2} \left\{h^2 ||u_k||_{H^3(\Omega(\mathcal{D}'''))} + ||u_k - U_k||_{H^o(\Omega(\mathcal{D}''))}\right\}$$

Proof. For any $w \in \mathring{M}(\mathcal{D}, \Pi(\mathcal{D}''))$ we have

$$(8.7) \qquad B(U - \Pi u, w) = B(u - \Pi u, w) = \sum_{\Delta \in \mathcal{D}''} B_\Delta(u - \Pi u, w)$$

where $\Pi u \in M(\mathcal{D})$ is such that $(\Pi u)(x) = u(x)$ for any $x \in \mathcal{R}(\mathcal{D})$, and B_Δ is the restriction of B to Δ.

On each $\Delta \in \mathcal{D}''$, $B_\Delta(u - \Pi u, w)$ is the sum of terms of the form

$$\int_\Delta \theta \frac{\partial}{\partial x_\ell} (u_j - \Pi u_j) \frac{\partial w_i}{\partial x_k} dx$$

for $i,j,k,\ell = 1,2$ and some constant θ. Consider first the terms

$$\int_\Delta \frac{\partial}{\partial x_k} (u_j - \Pi u_j) \frac{\partial w_i}{\partial x_k} dx \quad .$$

It is easy to see that for u_j a quadratic function the above expression is equal to zero. Therefore by standard argument we get

$$(8.8) \qquad \left| \int_\Delta \frac{\partial}{\partial x_k} (u_j - \Pi u_j) \frac{\partial w_i}{\partial x_k} dx \right| \le C h^2 |u_j|_{H^3(\Delta)} |w_i|_{H^1(\Delta)}$$

with C independent of Δ and u,w.

Let us consider now the term

$$(8.9) \qquad \int_\Delta \frac{\partial}{\partial x_k} (u_j - \Pi u_j) \frac{\partial w_i}{\partial x_\ell} dx, \quad k \ne \ell \quad .$$

Obviously we need only analyse the case $k = 1$, $\ell = 2$.

Using the notation as shown in Figure 8.1 we integrate by parts in (8.9)

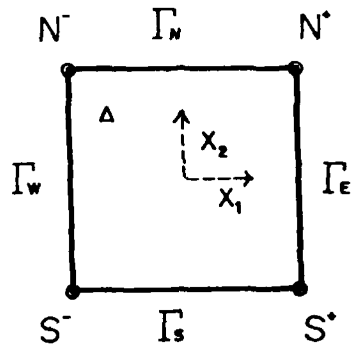


Figure 8.1. The notation of an element.

and get

$$\int_\Delta \frac{\partial}{\partial x_1} (u_j-\Pi u_j) \frac{\partial w_i}{\partial x_2}\, dx = \int_{\Gamma_N-\Gamma_S} \frac{\partial}{\partial x_1} (u_j-\Pi u_j) w_i dx_1 -$$

$$- \int_\Delta \frac{\partial^2}{\partial x_1 \partial x_2} (u_j-\Pi u_j) w_i dx = (u_j-\Pi u_j)w_i \Big|_{N^-}^{N^+} - \int_{\Gamma_N} (u_j-\Pi u_j) \frac{\partial w_i}{\partial x_1} dx_1$$

$$- (u_j-\Pi u_j)w_i \Big|_{S^-}^{S^+} + \int_{\Gamma_S} (u_j-\Pi u_j) \frac{\partial w_i}{\partial x_1} dx_1 - \int_\Delta \frac{\partial^2}{\partial x_1 \partial x_2} (u_j-\Pi u_j) w_i dx \quad .$$

Because for a quadratic function we have $\frac{\partial}{\partial x_1 \partial x_2} (g-\Pi g) = 0$ we conclude

$$(8.10) \qquad \left| \int_\Delta \frac{\partial}{\partial x_1} (u_j-\Pi u_j) \frac{\partial w_i}{\partial x_2} dx + \int_{\Gamma_N-\Gamma_S} (u_j-\Pi u_j) \frac{\partial w_i}{\partial x_1} dx_1 \right|$$

$$\leq Ch^2 |u|_{H^3(\Delta)} |w_i|_{H^1(\Delta)} \quad .$$

Realizing that $(u_j - \Pi u_j)$ is a continuous function on $\Omega(\mathcal{D}'')$ and $\frac{\partial w_i}{\partial x_1}$ is continuous on all horizontal edges of $\mathcal{D}''$ and $\frac{\partial w_i}{\partial x_1} = 0$ on the horizontal edges of $\Omega(\mathcal{D}'')$ we get by adding (8.10) over all $\Delta \in \mathcal{D}''$

(8.11) $$\left| \int_{\Omega(\mathcal{D}'')} \frac{\partial}{\partial x_k} (u_j - \Pi u_j) \frac{\partial w_i}{\partial x_i} dx \right| \leq Ch^2 |u_j|_{H^3(\Omega(\mathcal{D}''))} |w_i|_{H^1(\Omega(\mathcal{D}''))}$$

for $k \neq i$

(8.8), (8.11) and (8.7) yield

(8.12) $$|B(U-\Pi u, w)| \leq Ch^2 \left(\sum_{j=1}^{2} |u_j|_{H^3(\Omega(\mathcal{D}''))} \right) \left(\sum_{j=1}^{2} |w_j|_{H^1(\Omega(\mathcal{D}''))} \right) .$$

Applying now theorem (8.1) we get from (8.12)

(8.13) $$|U_j - u_j|_{H^1(\Omega(\mathcal{D}'''))} \leq C \sum_{j=1}^{2} \left\{ h^2 ||u_j||_{H^3(\Omega(\mathcal{D}''))} + \frac{1}{\rho(\Omega(\mathcal{D}''),\Omega(\mathcal{D}'''))} |U_j - \Pi u_j|_{H^0(\Omega(\mathcal{D}''))} \right\}$$

$$\leq \frac{C}{\rho(\Omega(\mathcal{D}'''),\Omega(\mathcal{D}''))} \sum_{j=1}^{2} \left\{ h^2 ||u_j||_{H^3(\Omega(\mathcal{D}''))} + ||U_j - u_j||_{H^0(\mathcal{D}(\Omega''))} \right\}$$

where we have used the fact that

(8.14) $$||U_j - \Pi u_j||_{H^0(\Omega(\mathcal{D}''))} \leq ||U_j - u_j||_{H^0(\Omega(\mathcal{D}''))} + ||u_j - \Pi u_j||_{H^0(\Omega(\mathcal{D}''))}$$

and $$||u_j - \Pi u_j||_{H^0(\Omega(\mathcal{D}''))} \leq Ch^2 ||u||_{H^2(\Omega(\mathcal{D}''))} .$$

Let us consider further $D_i(u_j-\Pi u_j)$ on $\Delta \in \mathcal{D}'''$. Refering to Figure 8.2

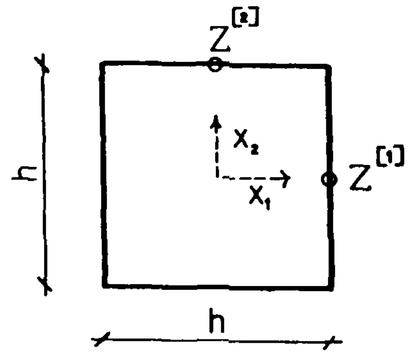


Figure 8.2. The notation of an element.

for any $w \in H^2(\Delta)$ we choose $\phi_i = J_i^\Delta w \in H_i(\Delta)$ so that

$$\phi_i(Z^{(i)}) = w(Z^{(i)})$$

where $Z^{(i)}$ are the midpoints of the particular sides as shown in Figure 8.2. For $u_j = g$ a quardatic function , we can easily check that

$$D_i(g-\Pi g) = J_i^\Lambda D_i g \quad .$$

And so by the usual arguments we have that for some $\phi_{j,i}^\Delta \in H_i(\Delta)$,

$$\text{(8.15)} \quad ||D_i(u_j-\Pi u_j)-\phi_{j,i}^\Delta||_{H^o(\Lambda)} \leq Ch^2|u_j|_{H^3(\Delta)} \quad .$$

Summing over all $\Delta \in \mathcal{D}'''$ we obtain

$$(8.16)\qquad \Big\|D_i(u_j-\Pi u_j) - \sum_{\Lambda\in\mathcal{D}'''} \phi^{\Lambda}_{j,i}\Big\|_{H^0(\Omega(\mathcal{D}'''))} \le Ch^2|u_j|_{H^3(\Omega(\mathcal{D}'''))} .$$

Combining now (8.16) and (8.13) we get the desired result.

LEMMA 8.4. Using the same notation as in Theorem 8.3 we get for $k \neq \ell$

$$|(D_\ell(u_j-U_j), D_k(u_i-U_i))_{H^0(\Omega(\mathcal{D}'''))}| \le C\Big\{\sum_{j=1}^{2} h\|u_j\|_{H^3(\Omega(\mathcal{D}''))} + \rho^{-1}\|u_j-U_j\|_{H^0(\Omega(\mathcal{D}''))}\Big\}\rho^{-1}\Big\{\sum_{j=1}^{2} h^2\|u_j\|_{H^3(\Omega(\mathcal{D}''))} + \|u_j-U_j\|_{H^0(\Omega(\mathcal{D}'''))}\Big\}$$

where $\rho = \rho(\Omega(\mathcal{D}'''), \Omega(\mathcal{D}''))$.

Proof. Write

$$E = \rho^{-1}(\Omega(\mathcal{D}'''), \Omega(\mathcal{D}''))\Big(\sum_{j=1}^{2} h^2\|u_j\|_{H^3(\Omega(\mathcal{D}''))} + \|u_j-U_j\|_{H^0(\Omega(\mathcal{D}''))}\Big) .$$

Now we have for any $\Lambda \in \mathcal{D}'''$

$$(8.17)\qquad (D_\ell(u_j-U_j), D_k(u_i-U_i))_{H^0(\Lambda)} = (\phi^{\Lambda}_{j,\ell} + D_\ell(u_j-U_j) - \phi^{\Lambda}_{j,\ell}, \phi^{\Lambda}_{i,k} + D_k(u_i-U_i) - \phi^{\Lambda}_{i,k})_{H^0(\Lambda)} = (\phi^{\Lambda}_{j,\ell}, \phi^{\Lambda}_{i,k})_{H^0(\Lambda)} + (D_\ell(u_j-U_j) - \phi^{\Lambda}_{j,\ell}, \phi^{\Lambda}_{i,k})_{H^0(\Lambda)}$$

$$+ (\phi^{\Lambda}_{j,\ell}, D_k(u_i-U_i) - \phi^{\Lambda}_{i,k})_{H^0(\Lambda)} + (D_\ell(u_j-U_j) - \phi^{\Lambda}_{j,\ell}, D_k(u_i-U_l) - \phi^{\Lambda}_{l,k})_{H^0(\Lambda)} .$$

It is easy to see that for $\ell \neq k$ we have $(\phi^{\Delta}_{j,\ell},\phi^{\Delta}_{i,h})_{H^o(\Delta)} = 0$. Summing (8.17) over all $\Delta \in \mathcal{D}'''$ and using Theorem 8.3 we get

$$(8.18)\quad |(D_\ell(u_j-U_j),D_k(u_i-U_i))|_{H^o(\Omega(\mathcal{D}'''))} \leq C((\Phi^2+\Phi(\sum_{i,j=1}^{2}\sum_{\Delta\in\mathcal{D}'''}|\phi^{\Delta}_{i,j}|^2_{H^o(\Delta)})^{1/2}) .$$

Using once more Theorem 8.3, we get

$$(8.19)\quad ||\sum_{\Delta\in\mathcal{D}'''}\phi^{\Delta}_{j,i}||_{H^o(\Omega(\mathcal{D}'''))} \leq C[\Phi+||D_i(u_j-U_j)||_{H^o(\Omega(\mathcal{D}'''))}] .$$

Using now Theorem 8.1 for $\xi = u-U$ we have

$$(8.20)\quad ||D_i(u_j-U_j)||_{H^o(\Omega(\mathcal{D}'''))} \leq C(\sum_{j=1}^{2}\{h||u_j||_{H^2(\Omega(\mathcal{D}''))} + \rho^{-1}||u_j-U_j||_{H^o(\Omega(\mathcal{D}''))})$$

and combining now (8.18), (8.19), and (8.20), we obtain the desired result.

<u>LEMMA 8.5.</u> <u>Let S be the square as in Figure 8.1 with side length h . Then for any</u>

$$f = ax_1 + bx_2 + cx_1x_2 + d$$

<u>we have</u>

$$(8.21)\quad \Big|\, ||ax_1||^2_{H^o(S)} - \frac{1}{12}h^2(f^2(N^+)+f^2(N^-)+f^2(S^-)+f^2(S^+))\Big|$$

$$\leq M||bx_2+cx_1x_2+d||^2_{H^o(S)}$$

with M independent of h, a, b, c, d.

Proof. By simple computation we get

$$||ax_1||^2_{H^o(S)} = a^2 \frac{1}{12} h^4$$

$$f^2(N^+)+f^2(N^-)+f^2(S^-)+f^2(S^+) = (a^2+b^2+\frac{c^2}{4} + 4d^2)h^2$$

and

$$h^4(b^2 + \frac{c^2}{4} + 4d^2) \leq M||bx_2 + cx_1x_2 + d||^2_{H^o(S)}$$

which immediately yields (8.21).

THEOREM 8.6. Using the same notation as in Theorem 8.2 we get

$$(8.22) \quad \left| \, ||D_i(u_j-U_j)||^2_{H^o(\Omega(\mathcal{D}'''))} - \psi^2_{i,j}(\mathcal{D}''') \right| \leq C \sum_{k=1}^{2} \{(\rho^{-1}h^3+\rho^{-2}h^4)||u_k||^2_{H^3(\Omega(\mathcal{D}''))} + \rho^{-2}||u_k-U_k||^2_{H^o(\Omega(\mathcal{D}''))} + (\rho^{-1}h+\rho^{-2}h^2)||u_k||_{H^3(\Omega(\mathcal{D}''))} ||u_k-U_k||_{H^o(\Omega(\mathcal{D}''))}\}$$

where

$$(8.23) \quad \psi^2_{i,j}(\mathcal{D}''') = \sum_{\Delta\in\mathcal{D}'''} \frac{h^2}{48} [J^2_{i,j}(\Delta)(N^-) + J^2_{i,j}(\Delta)(N^+) + J^2_{i,j}(\Delta)(S^-) + J^2_{i,j}(\Delta)(S^+)].$$

and $J_{i,j}(\Delta)(N^-)$ is the jump in $\frac{\partial U_j}{\partial x_i}$ with i^{th} coordinate direction at the point N^- of Δ (i.e. the value of the jump across w at N^-) and analogously for the other vertices.

Proof. We have

$$(8.24) \qquad D_i(u_j-U_j) = (D_iu_j-T_i^hU_j) + (T_i^hU_j-D_iU_j) \quad .$$

By Theorem 8.2 we get

$$(8.25) \quad ||D_iu_j - T_i^hU_j||_{H^0(\Omega(\mathcal{D}'''))} \leq C[\sum_{k=1}^{2} (h^2||u_k||_{H^3(\Omega(\mathcal{D}'))} +$$

$$+ [\rho(\Omega(\mathcal{D}'''),\Omega(\mathcal{D}''))]^{-1}||u_k-U_k||_{H^0(\Omega(\mathcal{D}'))}] \quad .$$

Theorem 8.3 yields

$$(8.26) \qquad ||D_i(u_j-U_j) - \sum_{\Delta\in\mathcal{D}'''} \phi^{\Delta}_{j,i}||_{H^0(\Omega(\mathcal{D}'''))} \leq C[\rho^{-1}(\Omega(\mathcal{D}'''),\Omega(\mathcal{D}'')) \sum_{k=1}^{2}$$

$$\{h^2||u_k||_{H^3(\Omega(\mathcal{D}''))} + ||u_k-U_k||_{H^0(\Omega(\mathcal{D}''))}\}] = E_1 \quad .$$

Combining (8.24), (8.25) and (8.26) we get

$$(8.27) \qquad ||T_i^hU_j-D_iU_j-\sum_{\Delta\in\mathcal{D}'''} \phi^{\Delta}_{j,i}||_{H^0(\Omega(\mathcal{D}'''))} \leq C\ E_1 \quad .$$

Simple computation shows that $T_i^hU_j-D_iU_j$ is bilinear on each $\Delta\in\mathcal{D}'''$. For sake of definitions and without loss of any generality let us suppose that $i = 1$.

Consider now $T_1^hU_j-D_1U_j$. Using notation shown in Figure 8.3 we have for $\xi \equiv (\xi_1,\xi_2)\in\Gamma_E$

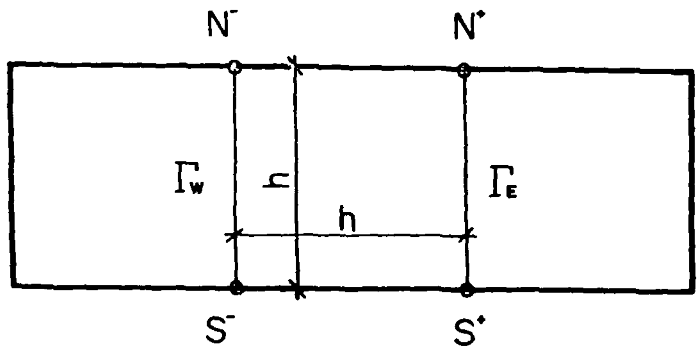


Figure 8.3. The scheme of the notation.

(8.28) $(T_1^h U_j - D_i U_j)(\xi) = 2h^{-1}[U_j(\xi_1+h,\xi_2) - U_j(\xi_1-h,\xi_2)] - h^{-1}[U_j(\xi)-U_j(\xi_1-h,\xi_2)] =$

$= \frac{1}{2} J_{1,j}(\xi)$.

For $\xi \in \Gamma_W$ we get analogously

(8.29) $(T_1^h U_j - D_i U_j)(\xi) = -\frac{1}{2} J_{1,j}(\xi)$

Let now $\tilde{\phi}^{\Delta}_{1,j}$ be the L_2 projection of $(T_1^h U_j - D_1 U_j)$ onto $H_1(\Delta)$. Then from (8.27) we get

(8.30) $||T_1^h U_j - D_1 U_j)||^2_{H^o(\Omega(\mathcal{D}'''))} = \sum_{\Delta \in \mathcal{D}'''} ||\tilde{\phi}^{\Delta}_{1,j}||^2_{H^o(\Delta)} + O(E_1^2)$.

Now it is easy to see that for any bilinear function f on Δ , $f = ax_1 + bx_2 + cx_1x_2 + d$ (with coordinate origin in the center of Δ) the L_2 projection of f onto $H_1(\Delta)$ is exactly ax_1 and so by lemma 8.5, (8.28) and (8.29) we obtain

$$(8.31)\qquad \sum_{\Lambda\in\mathcal{D}'''} \|\tilde{\phi}^{\Lambda}_{1,j}\|^2_{H^o(\Lambda)} = \frac{h^2}{48} \sum_{\Lambda\in\mathcal{D}'''} [J^2_{1,j}(N^-) + J^2_{1,j}(N^+) + J^2_{1,j}(S^-) + J_{1,j}(S^+)] +$$

$$O(\|T^h_1U_j - D_1U_j - \sum_{\Delta\in\mathcal{D}'''} \tilde{\phi}^{\Lambda}_{1,j}\|_{H^o(\Omega(\mathcal{D}'''))}) \quad .$$

Using now (8.27) and (8.30) we get

$$(8.32)\qquad \|T^h_1U_j - D_1U_j\|^2_{H^o(\Omega(\mathcal{D}''))} = \psi^2_{1,j}(\mathcal{D}''') + O(E^2_1) \quad .$$

Hence we have

$$(8.33)\qquad \|D_1(u_j - U_j)\|^2_{H^o(\Omega(\mathcal{D}'''))} \quad \|(T^h_1U_j - D_1U_j) + (D_1u_j - T^h_1U_j)\|_{H^o(\Omega(\mathcal{D}'''))} =$$

$$\|T^h_1U_j - D_1U_j\|^2_{H^o(\Omega(\mathcal{D}'''))} + \|D_1u_j - T^h_1U_j\|^2_{H^o(\Omega(\mathcal{D}'''))} +$$

$$+ \alpha\|D_1u_j - T^h_1U_j\|_{H^o(\Omega(\mathcal{D}'''))} \|T^h_1U_j - D_1U_j\|_{H^o(\Omega(\mathcal{D}'''))}$$

where $-2 < \alpha < 2$.

Further using (8.25) and (8.20)

$$(8.34)\qquad \|T^h_1U_j - D_1U_j\|_{H^o(\Omega(\mathcal{D}''))} \le \|D_1(u_j - U_j)\|_{H^o(\Omega(\mathcal{D}'''))} +$$

$$+ \|T^h_1U_j - D_1u_j\|_{H^o(\Omega(\mathcal{D}''))} \le C(E_1 + \sum_{j=1}^{2} h\|u_j\|_{H^2(\Omega(\mathcal{D}'))} + \rho^{-1}(\Omega(\mathcal{D}'''),\Omega(\mathcal{D}''))$$

$$\|u_j - U_j\|_{H^o(\Omega(\mathcal{D}'))} \quad .$$

Using now (8.32), (8.33), (8.34) and (8.25), we have

$$(8.35) \qquad ||D_1(u_j - U_j)||^2_{H^o(\mathcal{D}'')} = \psi^2_{1,j}(\mathcal{D}''') + O(E_1^2 + E_1(\sum_{j=1}^{2} h||u_j||_{H^3(\Omega(\mathcal{D}'))} +$$

$$+ \rho^{-1}(\Omega(\mathcal{D}'''),\Omega(\mathcal{D}''))||u_j - U_j||_{H^o(\Omega(\mathcal{D}'))})$$

and (8.35) yields almost immediately the theorem.

In section 6 and 7 we introduced various error indicators. See e.g. theorems 7.11 and 7.12. It is obvious that we can have many equivalent error indicators which would be simultaneously upper and lower ones. Theorem 8.6 enables us to design a special one, optimally suited to our purpose.

In (6.1) we introduced the basic bilinear form. Obviously we can write $B(u,v) = [Dv]A[Du]^T$ where

$$[Du] \equiv [D_1u_1, D_1u_2, D_2u_1, D_2u_2]$$

and analogously $D\underline{v}$.

The matrix A has then the form,

$$A \equiv \begin{bmatrix} \lambda+2\mu & 0 & 0 & \lambda \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \lambda & & & \lambda+2\mu \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Assume now that $\Delta \in \mathcal{D}'''$ as in theorem 8.2. Then define

$$\alpha^2(\Delta) = \sum_{j=1}^{4} \{[J_{1,1}, J_{1,2}]A_{11}[J_{1,1}, J_{1,2}]^T[a_j] + [J_{2,1}, J_{2,2}]A_{22}[J_{2,1}, J_{2,2}]^T$$

$$[a_j]\}$$

where a_j $j = 1,2,3,4$ are the four vertices of the element Δ.

Further let

$$\beta^2(\Delta) = \sum_{i=1}^{2} ||g_i - \hat{g}_i||^2_{H^o(\Delta)}$$

and then define the error indicator (which obviously is simultaneously an upper and lower one)

$$(8.36) \qquad \eta^2(\Delta) = \frac{h^2}{48}\alpha^2(\Delta) + h^2\cdot\gamma\beta^2(\Delta)$$

where $\gamma > 0$ is a constant which will be determined later. We mention here only that for smooth g we have $|\beta(\Delta)| \leq Ch$.

Now we have

THEOREM 8.7. Let the assumptions of Theorem 8.6 be satisfied. Then

$$(8.37) \qquad |||u-U|||^2_{\Omega(\mathcal{D}''')} = \sum_{\Delta\in\mathcal{D}'''} \frac{h^2}{48}\alpha^2(\Delta) + C[\sum_{k=1}^{2} \{(\rho^{-1}h^3+\rho^{-2}h^4)||u_k||^2_{H^3(\Omega(\mathcal{D}'))} +$$

$$+ \rho^{-2}||e||^2_{H^o} + (\rho^{-1}h+\rho^{-2}h^2)||u_k||_{H^3(\Omega(\mathcal{D}'))}||e||_{H^o(\mathcal{D}')}] +$$

$$+ \{\sum_{j=1}^{2} h||u_j||_{H^3(\Omega(\mathcal{D}'))} + \rho^{-1}||e||_{H^o(\Omega(\mathcal{D}'))}\}\{\sum_{j=1}^{2} h\,||u_j||_{H^3(\Omega(\mathcal{D}'))} +$$

$$+ ||e||_{H^o(\Omega(\mathcal{D}'))}\} .$$

Proof. We have for $u - U = e = (e_1, e_2)$

$$|||u-U|||^2_{\Omega(\mathcal{D}''')} =$$

$$\int_{\Omega(\mathcal{D}''')} \{(\lambda+2\mu)((D_1e_1)^2+(D_2e_2)^2) + \mu[(D_2e_1)^2+(D_1e_2)^2] + 2\mu[D_2e_1D_1e_2] + 2\lambda[D_1e_1D_2e_2]\}dx \quad .$$

Using Lemma 8.4 and Theorem 8.6 we get the desired result.

So far we have defined the error indicator $\eta(\Delta)$ by (8.36) for all $\Delta \in \mathcal{D}'''$. We will extend this definition to all elements Δ with $\Delta \in Q_\theta^k$ (see sections 2 and 6) where the coefficients are constant and $\Delta \cap \partial Q_\theta^k = \emptyset$. If Δ has an edge on ∂Q_θ^k then we shall define the indicator in terms of the jumps in $\sigma_{i,j}$, so that we obtain an upper and lower estimator. The detailed formulation and extensive numerical experience will be addressed in a forthcoming paper.

Here we will assume only

a) The indicator has the form (8.36) for any $\Delta \in Q_\theta^k$, $\Delta \cap \partial Q_\theta^k = \emptyset$ $(\gamma > 0)$ fixed but arbitrary.

b) All the meshes on K-meshes and the indicator leads to a simultaneous upper and lower estimator (under the assumptions spelled out in Section 7).

We will place some additional assumptions on the solution u and the meshes used, as listed below. Then we shall prove that the error estimator is asymptotically correct. By this we mean that

$$\frac{\mathcal{E}}{|||e|||} \to 1$$

as $|||e||| \to 0$.

We will assume that there is a sequence of K-meshes $\mathcal{D}_h$, $h(\mathcal{D}_h) = h$, $h \to 0$ with K independent of h .

c) The mesh is equilibrated in the sense that $\max_{\Delta\in\mathcal{D}_h} \eta(\Delta) / \min_{\Delta\in\mathcal{D}_h} \eta(\Delta) \le C$ with C independent of h.

d) $|||e|||_{\Omega} \ge Ch$ with $C > 0$ independent of h.

e) There exists $s > 0$ and C, both independent of h, such that

$$|||e|||_{H^o(\Omega)} \le Ch^s |||e|||$$

f) For each $\varepsilon > 0$, $h > 0$, there exists meshes $\mathcal{D}'_{h,\varepsilon,i} > \mathcal{D}''_{h,\varepsilon,i} > \mathcal{D}'''_{h,\varepsilon,i}$ $i = 1,2\ldots m(h,\varepsilon)$ such that

i) the bilinear form has constant coefficients on $\Omega(\mathcal{D}'_{h,\varepsilon,i})$

ii) $\mathcal{D}''_{h,\varepsilon,i}$ are uniform and $\mathcal{D}'_{h,\varepsilon,i}$ are of uniform size

iii) $\Omega(\mathcal{D}'''_{h,\varepsilon,i}) \cap \Omega(\mathcal{D}'''_{h,\varepsilon,j}) = \emptyset$ for all $i \ne j$.

iv) $\rho(\Omega(\mathcal{D}''_{h,\varepsilon,i}), \Omega(\mathcal{D}'''_{h,\varepsilon,i})) \ge Ch^\theta$ with C and θ independent of h and ε, $0 < \theta < S$.

g) Denote $R'_{h,\varepsilon} = \bigcup_{i=1}^{m(h,\varepsilon)} \Omega(\mathcal{D}'_{h,\varepsilon,i})$ and analogously $R''_{h,\varepsilon}$ and $R'''_{h,\varepsilon}$ and we shall assume that

i) $||u||_{H^3(R'_{h,\varepsilon})} \le \chi(\varepsilon)$

ii) $||g||_{H^1(R'_{h,\varepsilon})} \le \chi(\varepsilon)$

h) Let $S'''_{h,\varepsilon} = \Omega - \bigcup_{e=1}^{m(\varepsilon,h)} \Omega(\mathcal{D}'''_{h,\varepsilon,i})$ and $S'''^{\,\delta(h)}_{h,\varepsilon}$ be the δ neighborhood of $S'''_{h,\varepsilon}$ with $\delta = h^\theta$ $0 < \theta < s$.

Denote now

$$\mathcal{J}_{h,\varepsilon} = \{\Delta \mid \Delta \cap S'''^{\,\delta(h)}_{h,\varepsilon} \ne \emptyset\},$$

and we will assume that

$$\sigma(\tilde{\mathcal{D}}_{h,\varepsilon}) \le \varepsilon\sigma(\mathcal{D}_h)$$

where $\sigma(\mathcal{D}_h)$ respective $\sigma(\tilde{\mathcal{D}}_{h,\varepsilon})$ denotes the number of elements contained in $\mathcal{D}_h$ respective $\tilde{\mathcal{D}}_{h,\varepsilon}$.

Now we have

THEOREM 8.8. Suppose that the assumptions listed above are satisfied, then $\varepsilon_o > 0$ such that for $\varepsilon < \varepsilon_o$ and $h \le h_o(\varepsilon)$

$$\left| \frac{E}{|||e|||_\Omega} - 1 \right| \le M\varepsilon$$

where M is independent of h and ε .

Proof. For every $\mathcal{D}'''_{h,\varepsilon,i}$ we can write by theorem 8.7 and property f) and g)

$$\sum_{i=1}^{m(h,\varepsilon)} |||e|||^2_{\Omega(\mathcal{D}'''_{h,\varepsilon,i})} = \sum_{\Delta \in \bigcup_{i=1}^{m} \mathcal{D}''_{h,\varepsilon,i}} \eta^2(\Delta) + z - \hat{z}$$

where

$$|z| \le C[(h^{3-\theta}+h^{4-2\theta})\chi^2(\varepsilon) + h^{-2\theta+2s}|||e|||^2_\Omega + (h^{1+s-\theta}+h^{2+s-2\theta})\chi(\varepsilon)|||e|||_\Omega +$$

$$+ (h^{1+s-\theta}\chi(\varepsilon) + h^{2s-2\theta}|||e|||_\Omega)(\chi(\varepsilon)h^{2-s}+|||e|||)]$$

and

$$|z| = \sum_{\Lambda \in \bigcup_{i=1}^{m} \mathcal{D}'''_{h,\varepsilon,i}} \gamma h^2 \sum_{i=1}^{2} ||g_i - \bar{g}_i||^2_{H^0(\Lambda)}$$

Taking into consideration the property d) we get

$$|z| \le C|||e|||^2_\Omega \, (h^{1-\theta}+h^{2-2s})\chi^2(\varepsilon) + h^{2(s-\theta)} + (h^{s-\theta}+h^{1+s-2\theta})\chi(\varepsilon) +$$

$$+ (h^{1-\theta}\chi^2(\varepsilon) + h^{1+s-2\theta}\chi(\varepsilon) + h^{s-\theta}\chi(\varepsilon) + h^{2s-2\theta}]$$

We have

$$||g_i - \bar{g}_i||^2_{H^0(\Lambda)} \le Ch^2||g_i||^2_{H^1(\Lambda)}$$

and therefore

$$|\hat{z}| \le Ch^4 \sum_{i=1}^{2} ||g_i||^2_{\Omega(\mathcal{D}'''_{h,\varepsilon,i})} \le Ch^2||e||^2\chi(\varepsilon) \quad .$$

Hence

$$|z| + |\hat{z}| \le \varepsilon|||e|||^2_\Omega$$

for $h < h_i(\varepsilon)$.

We have further

$$\sum_{\Delta\in\mathcal{D}_h} \eta^2(\Delta) = \sum_{\Delta\in \bigcup_{i=1}^{m} \mathcal{D}'''_{h,\varepsilon,i}} \eta^2(\Delta) + \sum_{\substack{\Delta\in\mathcal{D}_h \\ \Delta\notin \bigcup_{i=1}^{m} \mathcal{D}'''_{h,\varepsilon,i}}} \eta^2(\Delta)$$

and by property (h) we get

$$\sum_{\substack{\Delta\in\tilde{\mathcal{D}}_h \\ \Delta\notin \bigcup_{i=1}^{m} \mathcal{D}'''_{h,\varepsilon,i}}} \eta^2(\Delta) \le \sigma(\tilde{\mathcal{D}}_{h,\varepsilon})\eta^2_{max}(\Delta) \le \varepsilon\sigma(\mathcal{D}_h)\eta^2_{max} \le \varepsilon \sum_{\Delta\in\mathcal{D}_h} \eta^2(\Delta) \frac{\eta^2_{max}}{\eta^2_{min}}$$

$$\le C\varepsilon \sum_{\Delta\in\mathcal{D}_h} \eta^2(\Delta)$$

when using property c) .

So we have for sufficiently small ε

$$(1-\varepsilon M) \sum_{\Delta\in\mathcal{D}_h} \eta^2(\Delta) \le \sum_{\Delta\in \bigcup_{i=1}^{m} \mathcal{D}'''_{h,\varepsilon,i}} \eta^2(\Delta) \le \sum_{\Delta\in\mathcal{D}_h} \eta^2(\Delta) \quad .$$

Using further the [illegible]act that the estimator is an upper and lower estimator we get

$$(8.39) \qquad \sum_{i=1}^{m(h,\varepsilon)} |||e|||^2_{\Omega(\mathcal{D}'''_{h,\varepsilon,i})} = E^2(1+\beta(h,\varepsilon)\varepsilon)$$

[illegible] $(h, \cdot)| < M_o$, M_o independent of h and ε .

[illegible]iously we have

$$[illegible] = \sum_{i=1}^{m(h,\varepsilon)} |||e|||^2_{\Omega(\mathcal{D}'''_{h,\varepsilon,i})} + |||e|||^2_{S'''_{h,\varepsilon}} \quad .$$

Let us analyze now $|||e|||^2_{S'''_{h,\varepsilon}}$. It is easy to see that there exists a function ψ_h such that

i) $0 \le \psi_h \le 1$ on Ω

ii) $|D^1\psi_h| \le \frac{C}{\delta(h)} \le Ch^{-\theta}$

iii) $\psi_h = 1$ on $S'''_{h,\varepsilon}$

$\psi_h = 0$ on $\Omega - S'''^{\,\delta(h)}_{h,\varepsilon}$.

Obviously now

$$(8.40) \qquad |||e|||^2_{S'''_{h,\varepsilon}} \le B(e\psi_h, e\psi_h)$$

A typical term of $B(e\,\psi_h, e\psi_h)$ is

$$\int_\Omega A\; D^i(e_j\psi_h)D^k(e_\ell\psi_h)dx$$

where A is piecewise constant.

We have now

$$D^i(e_j\psi_h)D^k(e_\ell\psi_h) = D^i e_j D^i(\psi_h^2 e_\ell) + z^*$$

where

$$z^* = \psi_h D^k\psi_h e_\ell D^i e_j + \psi_h D^i\psi_h e_j D^k e_\ell + e_j e_\ell D^i\psi_h D^k\psi_h - 2\psi_h D^k\psi_h D^i e_j e_\ell \quad .$$

Hence

$$\left| \int_\Omega Az^*dx \right| \le C|||e|||^2_\Omega [h^{s-\theta}+h^{2s-2\theta}]$$

and so we have

(8.41) $\quad B(e\psi_h, e\psi_h) = B(e, e\psi_h^2) + R$

where

(8.42) $\quad R \le \varepsilon |||u|||^2$

if $h \le h_o(\varepsilon)$.

Let us analyze now the term $B(e, e\psi_h^2)$. Obviously $\operatorname{supp} e\psi_h^2 \subset \Omega(\tilde{\mathcal{D}}_{h,\varepsilon})$. Using theorem 5.9 and the remark to it and lemma 4.1, we can find $w \in M(\mathcal{D})$, $\operatorname{supp} w \subset \Omega(\tilde{\tilde{\mathcal{D}}}_{h,\varepsilon})$ such that

$$\sum_{P \in R(\mathcal{D})} ||v_p(e\psi_h^2 - w)||^2_{H^1(\Omega)} \le C||e\psi_h^2||^2_{H^1(\Omega)}$$

and $\sigma(\tilde{\tilde{\mathcal{D}}}_{h,\varepsilon}) \le \bar{K}\sigma(\tilde{\tilde{\mathcal{D}}}_{h,\varepsilon})$.

Denote further

$$R^*_{h,\varepsilon} = \{P \in R(\mathcal{D}_h) \,|\, \omega_p \cap \tilde{\Omega}(\mathcal{D}_{h,\varepsilon}) = \emptyset\} .$$

Then repeating arguments of theorem 6.1 we get

(8.43) $$|B(e, \psi_h^2 e)| \le C\left[\sum_{P \in R^*_{h,\varepsilon}} |||\eta_p|||^2\right]^{1/2} |||\psi_h^2 e||| .$$

Denote now

$$\hat{\mathcal{D}}_{h,\varepsilon} = \{\Delta | \Delta \subset \omega_p,\ P \in \mathcal{R}^*_{h,\varepsilon}\} \quad .$$

Using lemma 4.1 we easily get

$$(8.44) \qquad \sigma(\hat{\mathcal{D}}_{h,\varepsilon}) \le \bar{K}\sigma(\tilde{\tilde{\mathcal{D}}}_{h,\varepsilon}) \le K_o\sigma(\tilde{\mathcal{D}}_{h,\varepsilon})$$

and

$$(8.45) \qquad \sum_{P\in\mathcal{R}^*_{h,\varepsilon}} |||\eta_p|||^2 \le C \sum_{\Delta\in\hat{\mathcal{D}}_{h,\varepsilon}} \eta^2(\Delta) \quad .$$

In fact if on ω_p the bilinear form has constant coefficients and $\omega_p \cap \partial\Omega = \emptyset$ then lemma 7.5 shows that $|||\eta_p|||$ can be estimated from above by the suitable turn of the error indicators. In the other cases we first use lemma 7.3 and lemma 7.5 and the same argument as used in theorem 7.8. Therefore we get

$$|B(e,\psi_h^2 e)| \le C[\sum_{\Delta\in\hat{\mathcal{D}}_{h,\varepsilon}} \eta^2(\Delta)]^{1/2} ||\psi_h^2 e||_\Omega \quad .$$

Now

$$|||\psi_h^2 e|||_\Omega \le C||\psi_h^2 e||_{H^1(\Omega)} \le C[||\psi_h^2 e||^2_{H^1(S'''_{h,\varepsilon})} + ||\psi_h^2 e||^2_{H^1(S^{\delta(h)}_{h,\varepsilon} - S_{h,\varepsilon})} +$$

$$+ ||\psi_h^2 e||^2_{H^1(\Omega - S'''^{\,\delta(h)}_{h,\varepsilon})}]$$

because $\psi_h = 1$ on $S'''_{h,\varepsilon}$ and $\psi_h = 0$ on $\Omega - S'''^{\,\delta(h)}_{h,\varepsilon}$ we get

$$(8.46) \qquad |||\psi_h^2 e|||_\Omega \le C(||\psi_h e||^2_{H^1(S'''_{h,\varepsilon})} + ||\psi_h^2 e||^2_{H^1(S'''^{\,\delta(h)}_{h,\varepsilon} - S'''_{h,\varepsilon})}) \quad .$$

Consider now

$$(8.47)\quad ||\psi_h^2 e||_{H^1(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} \leq ||\psi_h^2 e||_{H^o(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} + |\psi_h^2 e|_{H^1(S'''^{\delta(h)}_{h,\varepsilon}-S''_{h,\varepsilon})}$$

$$\leq ||\psi_h e||_{H^o(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} + |\psi_h^2 e|_{H^1(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} \quad .$$

Now

$$(8.48)\quad |\psi_h^2 e|_{H^1(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} = \sum_{i=1}^{2}[||D^i\psi_h\psi_h e||_{H^o(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} +$$

$$+ ||\psi_h D^i(\psi_h e)||_{H^o(S'''^{(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})}]$$

and

$$(8.49)\quad ||\psi_h D^i(\psi_h e)||_{H^o(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} \leq ||D^i(\psi_h e)||_{H^o(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})}$$

$$\leq ||\psi_h e||_{H^1(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})}$$

$$(8.50)\quad ||(D^i\psi_h)\psi_h e||_{H^o(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} \leq Ch^{-\theta}||\psi_h e||_{H^o(S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} \quad .$$

For $x \in S'''^{\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon}$ we have

$$\mathrm{dist}(x,\Omega-S'''^{\,\delta(h)}_{h,\varepsilon}) \leq h^{\theta}$$

and so

$$||\psi_h e||_{H^o(S'''^{\,\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} \leq Ch^{\theta}|\psi_h e|_{H^1(S'''^{\,\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})}$$

because $\psi_h e = 0$ on $\Omega - S'''^{\,o(h)}_{h,\varepsilon}$.

So we have from (8.48), (8.49) and (8.50),

$$|\psi_h^2 e|^2_{H^1(S'''^{\,\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} \leq C|\psi_h e|^2_{H^1(S'''^{\,\delta(h)}_{h,\varepsilon}-S'''_{h,\varepsilon})} .$$

Recalling (8.46), (8.47), we get

$$|||\psi_h e|||^2_{\Omega} \leq C||\psi_h e||_{H^1(\Omega)}$$

and therefore

$$(8.51) \quad B(e,\psi_h^2 e) \leq C\,\sigma(\hat{\mathcal{D}}_{h,\varepsilon})^{1/2}\eta_{max}||\psi_h e||_{H^1(\Omega)} \leq C\varepsilon^{1/2}(\sigma(\mathcal{D}_h))^{1/2}\eta_{max}||\eta_h e||_{H^1(\Omega)}$$

$$\leq C\varepsilon^{1/2}\left[\sum_{\Delta\in\mathcal{D}_h}\eta^2(\Delta)\right]^{1/2}||\psi_h e||_{H^1(\Omega)} \leq C\varepsilon^{1/2}||e||_{H^1(\Omega)}||\psi_h e||_{H^1(\Omega)}$$

where we used the fact that the mesh is equilibrated in the sense of the assumption c) above and that the estimator is an upper and lower one.

Returning to (8.41)

$$||\psi_h e||^2_{H^1(\Omega)} \le B(\psi_h, \psi_h e) \le C\varepsilon^{1/2}(||e||^2_{H^1(\Omega)} + ||e||_{H^1(\Omega)} ||\psi_h e||_{H^1(\Omega)})$$

and so

$$||\psi_h e||^2_{H^1(\Omega)} \le C\varepsilon ||e||^2_{H^1(\Omega)}$$

i.e.

$$B(\psi_h e, \psi_h e) \le C\varepsilon ||e||^2_{H^1(\Omega)}$$

and so by (8.40)

$$|||e|||^2_{S'''_{h,\varepsilon}} \le C\varepsilon ||e||^2_{H^1(\Omega)}$$

and the result follows.

The theorem 8.8 obviously shows that the error estimator is asymptotically correct. This property is achieved for any $\gamma > 0$. Let us discuss now the selection of γ . Obviously when the finite element solution is identically zero then the term in the error indicator associated with the jumps on the edges will disappear and the error indicator consists only of the "volume" integral

$$\gamma(\text{diam } \Delta)^2 \sum_{e=1}^{2} ||g_i - \bar{g}_i||^2_{H^0(\Delta)} .$$

Assume now that we have a uniform mesh in $\mathbb{R}^2$ and the exact solution is periodic

$$u_1 = [\sin \frac{2\pi}{h} x_1 \sin \frac{2\pi}{h} x_2]C_1$$

$$u_2 = [\sin \frac{2\pi}{h} x_1 \sin \frac{2\pi}{h} x_2]C_2 \quad .$$

Then it is easy to see that the finite element solution is identically zero. This leads to the choice

$$(8.51) \qquad \gamma = \frac{1}{2\pi} \frac{\lambda+3\mu}{(\lambda+2\mu)^2+\lambda^2}$$

As said above, the choice of γ is rather arbitrary and (8.47) is one of the possibilities.

We have assumed many very particular properties of the meshes and solutions, in the theorem 8.8. The problem arises whether these conditions can be satisfied. Practically we create the meshes in an adaptive mode. The experience has shown that the meshes which are adaptively constructed have roughly these properties, and that the effectivity index, $\frac{\mathcal{E}}{|||e|||}$ seems to converge to 1 quicker than the theorem 8.8 supports.

For uniform meshes in $\mathbb{R}^2$ and smooth solutions it can be shown that

$$|||e|||_\Omega = \mathcal{E}(1+0(\mathcal{E}^2))$$

(see [11]) and adaptively created meshes seem to lead to the same behaviour.

In the next section we will discuss some concrete illustrative examples pertinent to these questions.

9. THE FEARS PROGRAM

Based on the theory explained above, the program FEARS (Finite Element Adaptive Research Solver) was developed. FEARS is a fully adaptive program solving a system of two elliptic equations and produces the error estimation (in various norms) together with the numerical solution of the given partial differential equation. The admissible domain is a union of curvilinear rectangles. The adaptive approach is based on equilibration of the error indicators. The description and experimentation with FEARS will be reported elsewhere.

In this paper we are using FEARS as an illustration of the developed theory. We will discuss here two examples. In both, we are concerned with the (plane strain) elasticity problem. We assume that $E = 1$ (E is the Young's elasticity modulus) and $\nu = .3$ (ν is the Poission ratio).

Example 1. The elasticity problem on the square with displacements prescribed on the boundary. The data are shown in Figure 9.1. It is easy to see that the solution belongs to the space $H^{2-\varepsilon}(\Omega)$ ($\varepsilon > 0$, arbitrary).

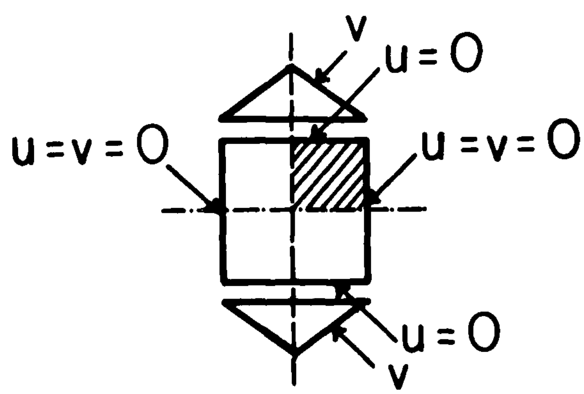


Figure 9.1. The data of example 1.

Example 2. The elasticity problem on the rectangle with mixed boundary conditions. The data are shown in Figure 9.2. The solution is of the "stamp" type

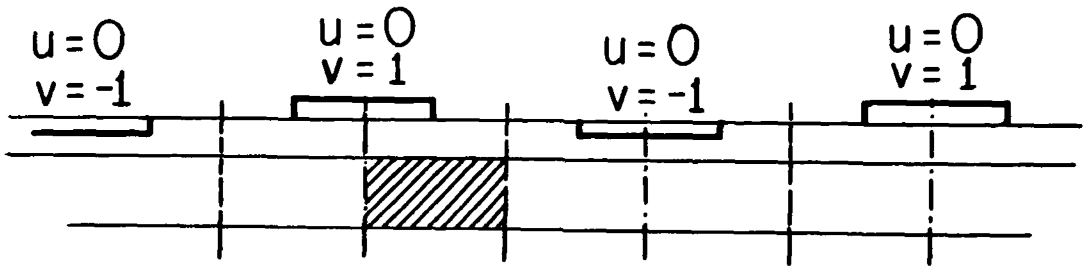


Figure 9.2. The data of example 2.

and the singularities are of the type described in [12] . Solution belong to the space $H^{3/2-\varepsilon}(\Omega)$, $\varepsilon > 0$ arbitrary.

In both cases the exact solution is not known, nevertheless by now elaborate computations we estimated with sufficient accuracy the exact energy of the solution. This gives the possibility to compute the (exact) energy norm of the error and compare it with the estimator.

Example 1. Because of the obvious symmetries of the solution, we can compute the solution only on the quarter of the original square applying boundary conditions shown in Figure 9.3.

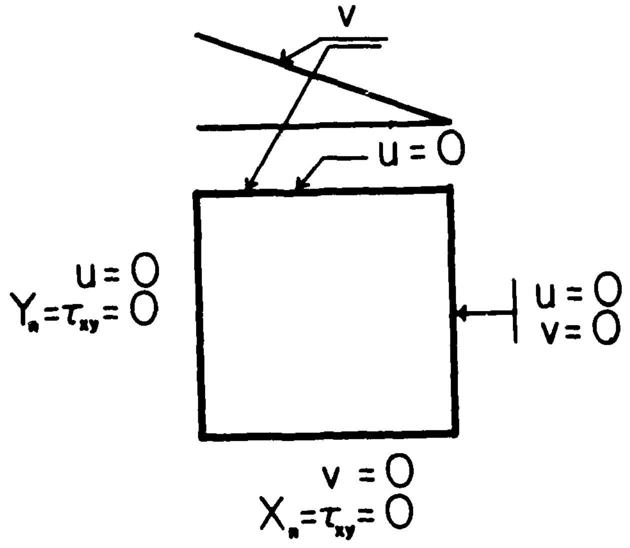


Figure 9.3. The boundary conditions for example 1.

FEARS constructs adaptively the meshes by equilibrating the error indicators.

Figure 9.4a,b,c,d,e,f,g show the sequence of constructed meshes. We see that the sequence of meshes satisfies the assumptions made in section 8.

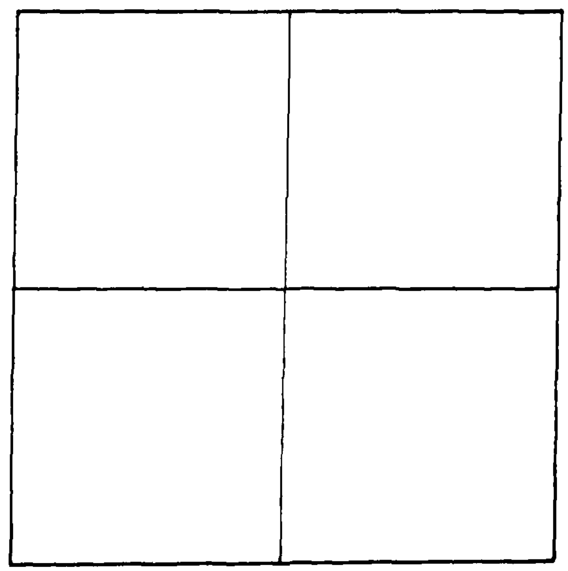
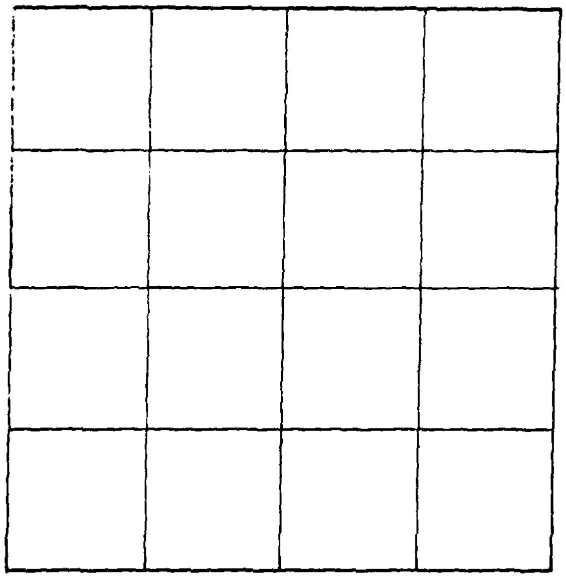
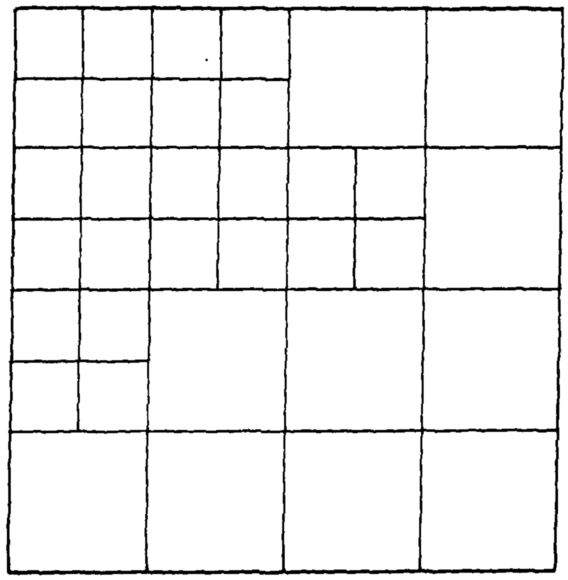
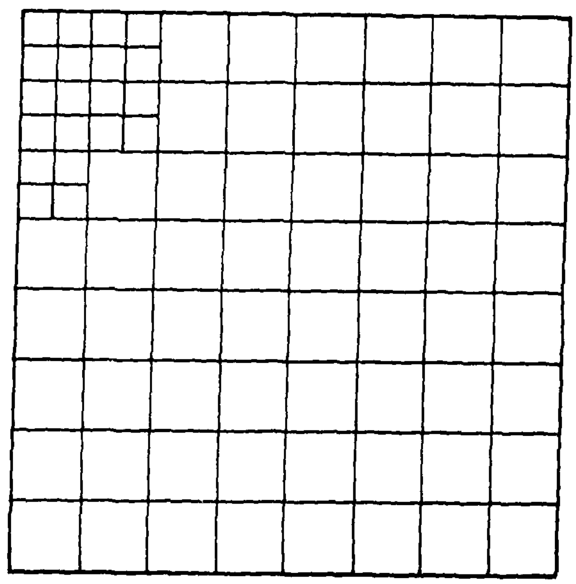
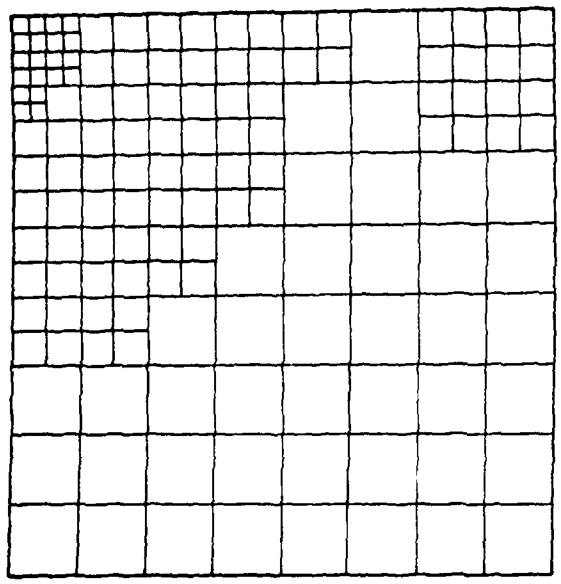


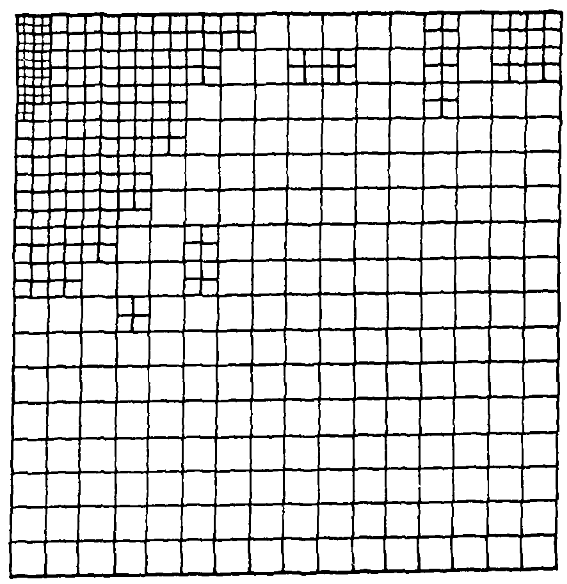
Figure 9.4. The sequence of adaptively constructed meshes for example 1.











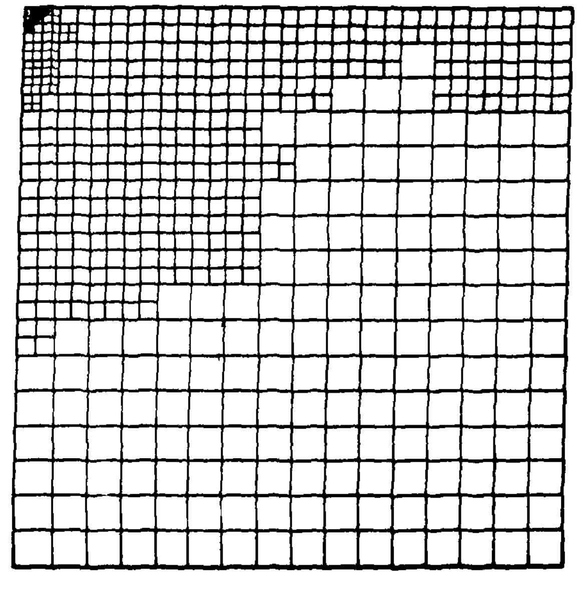


Figure 9.5 shows the dependence of the energy norm of the error on the number of the elements N. The norm at the error is measured in percent of the energy norm of the solution ($|||u|||$). Because the solution belongs to $H^{2-\varepsilon}(\Omega)$, the rate of convergence is $N^{-\frac{1}{2}}$ (or more precisely $N^{-\frac{1}{2}+\varepsilon}$) for the uniform mesh as for the adaptive one. The rate $N^{-\frac{1}{2}}$ is the maximal possible rate because of the type of elements used.

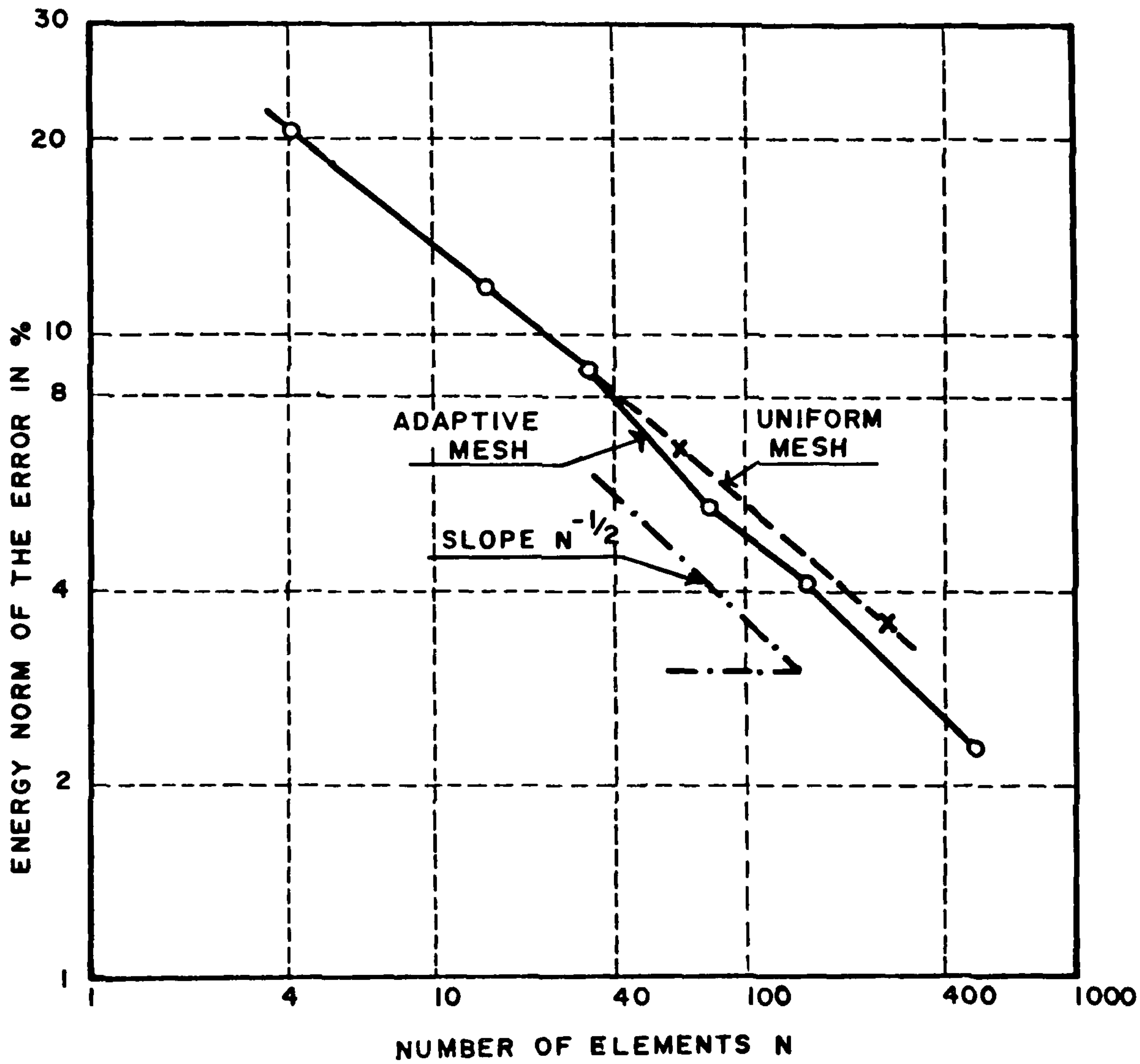


Figure 9.5. The energy norm of the error.

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Figure 9.5 shows the slope $N^{-\frac{1}{2}}$ in addition to the error behavior of the uniform and adaptive mesh.

The Figure 9.6 shows the effectivity index θ respect $1 - \theta$ as a function of the number of elements. We see that for the accuracy in the range of 5-10%, the effectivity index is quite acceptable from a practical point of view. We also see that $\theta \rightarrow 1$ converges with a higher rate than the error itself.

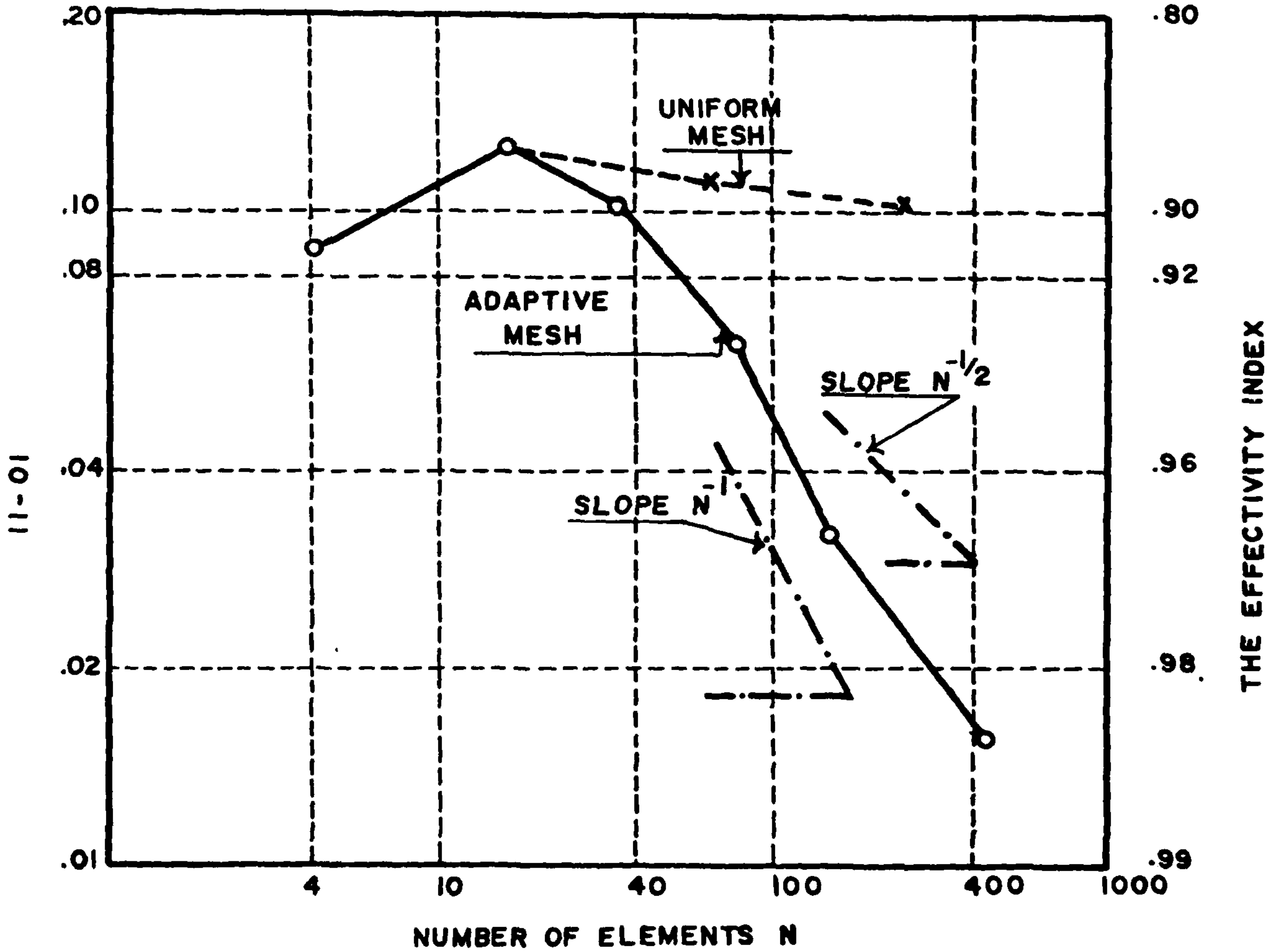


Figure 9.6. The effectivity index -- Example 1.

We also see that the quality of the error estimator is better for the adaptively constructed meshes than for the uniform ones. This is likely the consequence of the equilibration of the error indicators which is essential in our theory.

Example 2. Because of the obvious symmetries of the solution, we can restrict the computation to the domain on a boundary condition shown in Figure 9.7.

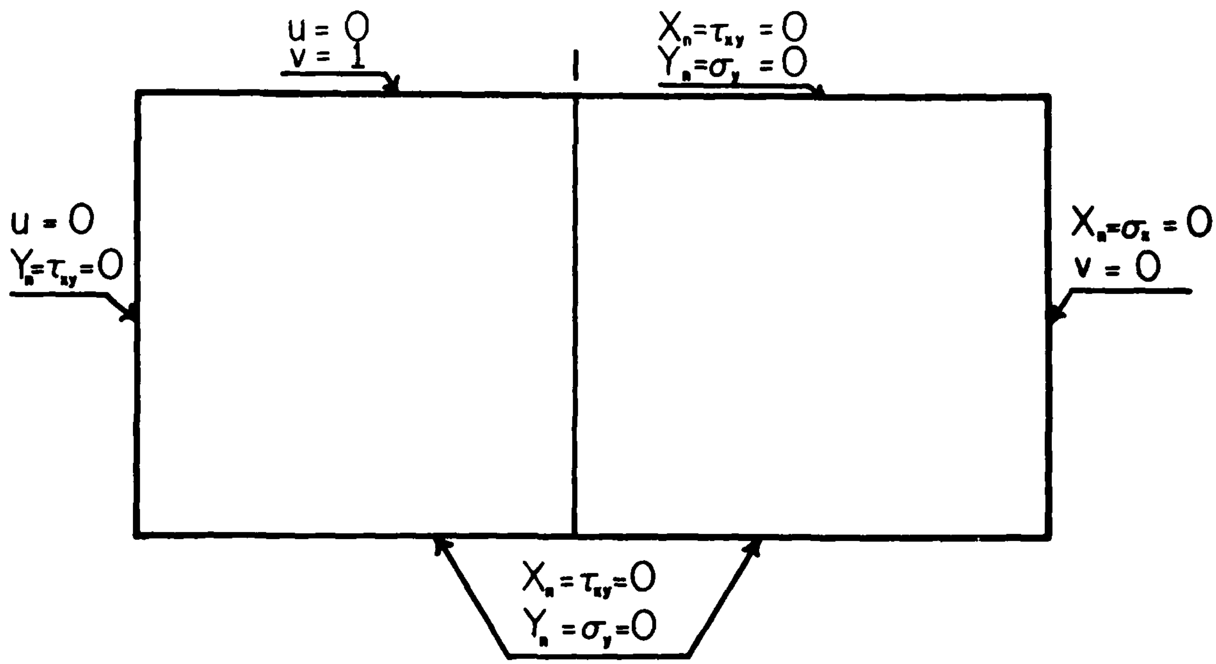


Figure 9.7. The boundary condition for example 1.

Figure 9.8a, b, c, d, e, f, g, h, i shows the sequence of the meshes constructed by FEARS. Once more we see that the assumption about the mesh made in section 8 is essentially satisfied.

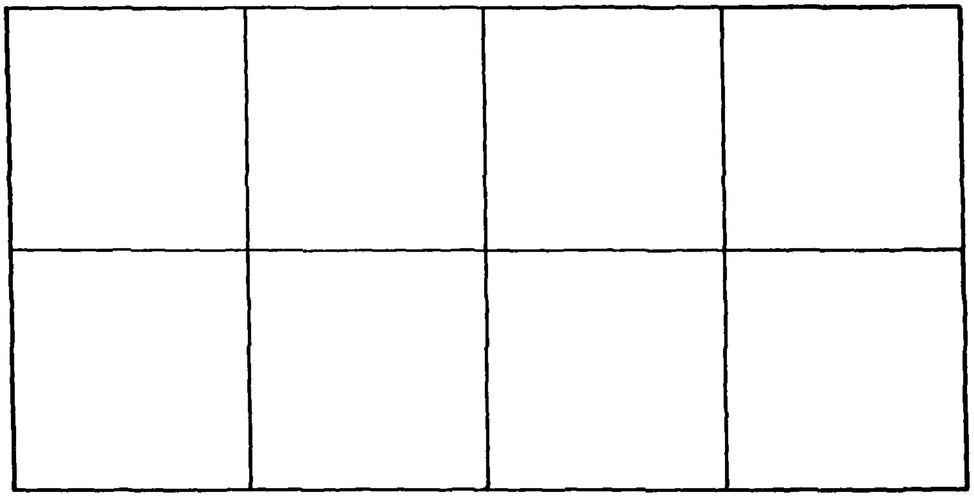
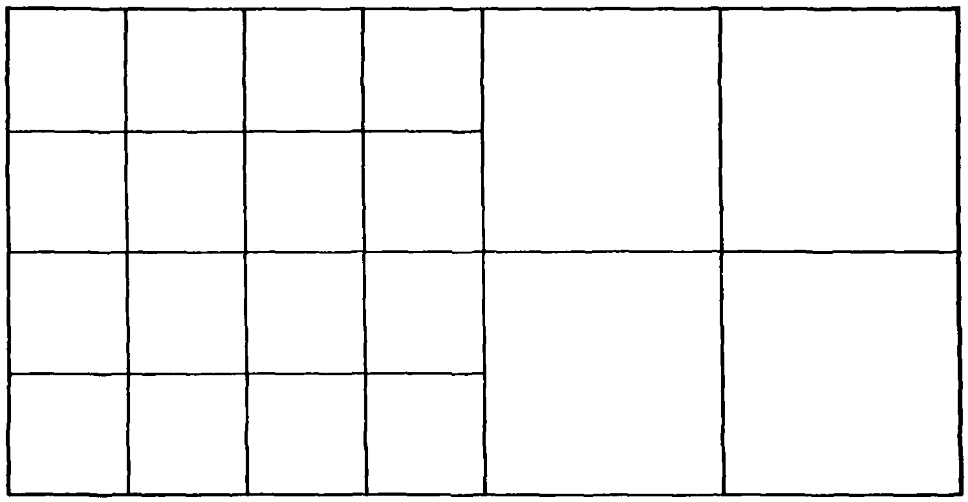
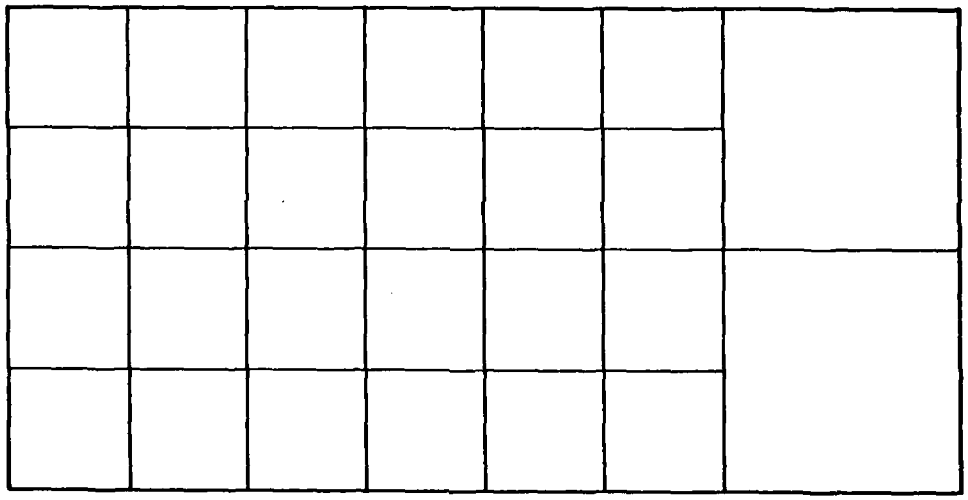
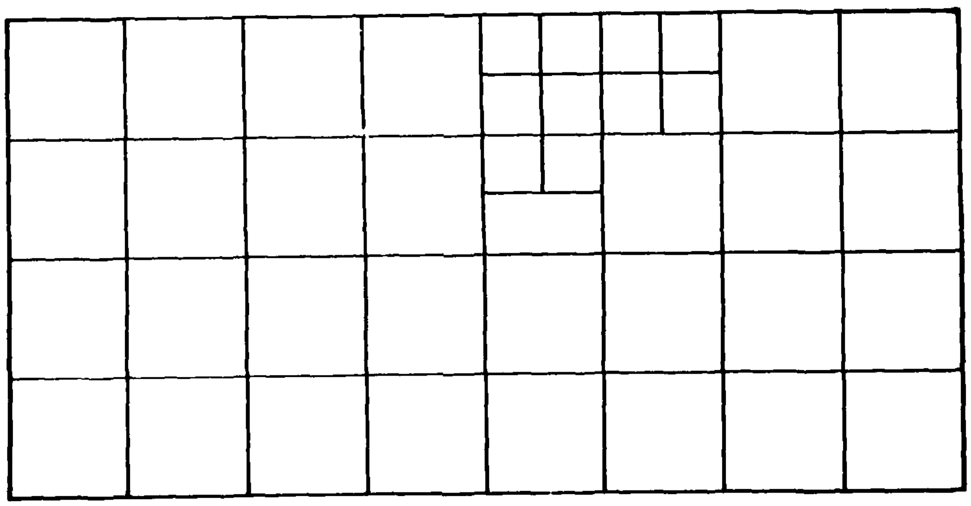


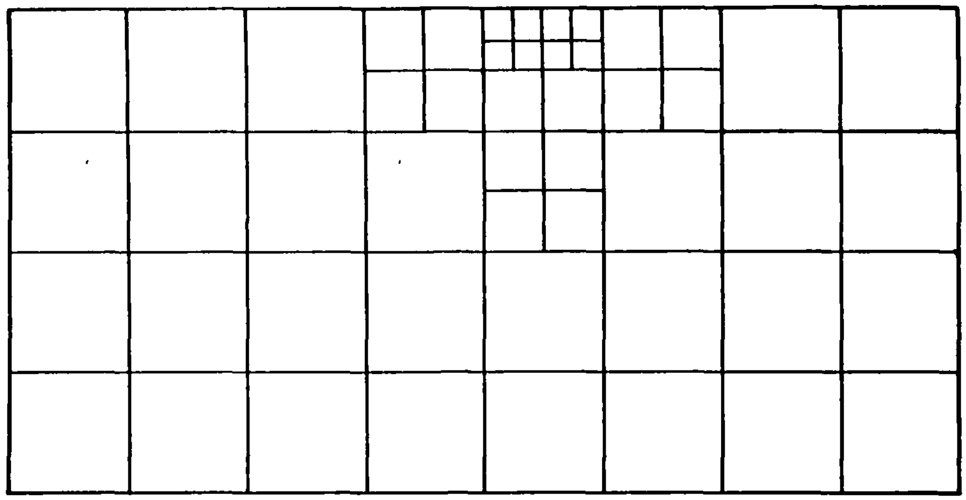
Figure 9.8. The sequence of adaptively constructed meshes for example 2.

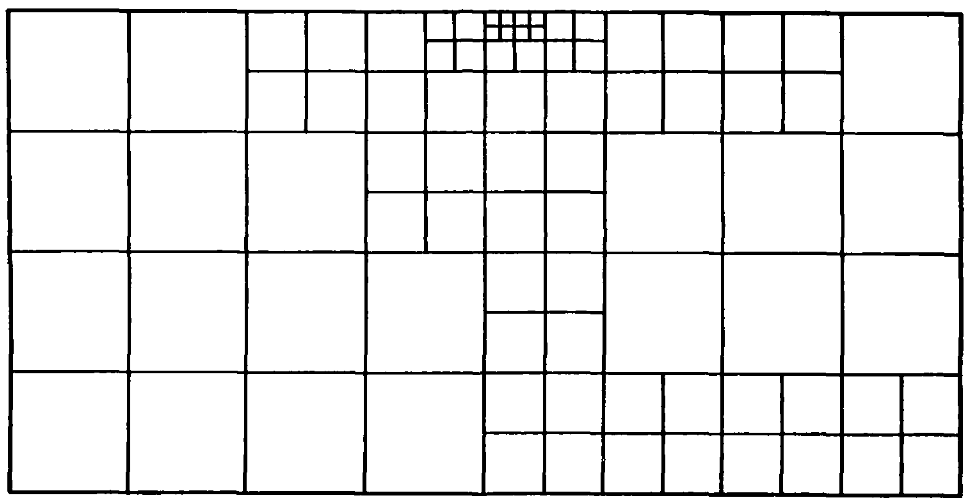
Figure 9.9 shows the behavior of the energy norm of the error. Because the solution belongs to $H^{3/2-\varepsilon}$ ($\varepsilon > 0$ arbitrary) and u $H^{3/2}$ the rate of convergence of the uniform mesh is $N^{-\frac{1}{4}}$. This is in complete agreement with the data shown in Figure 9.9. The adaptive mesh gives the rate of convergence $N^{-\frac{1}{2}}$ which is the maximal possible rate for the smooth solution. We see that the adaptive mesh

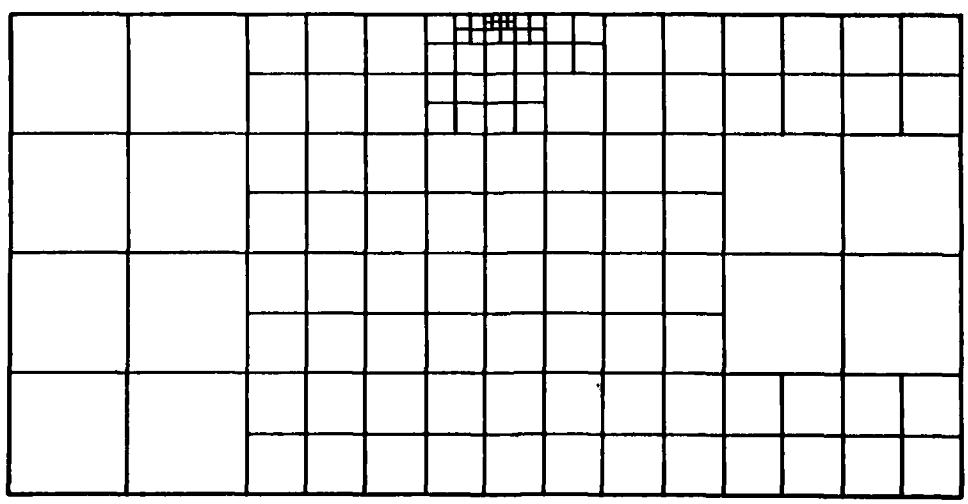


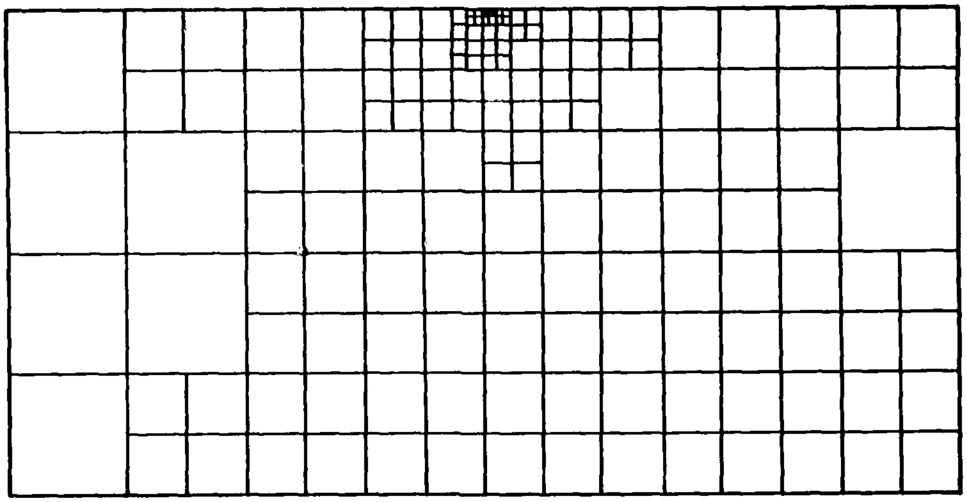


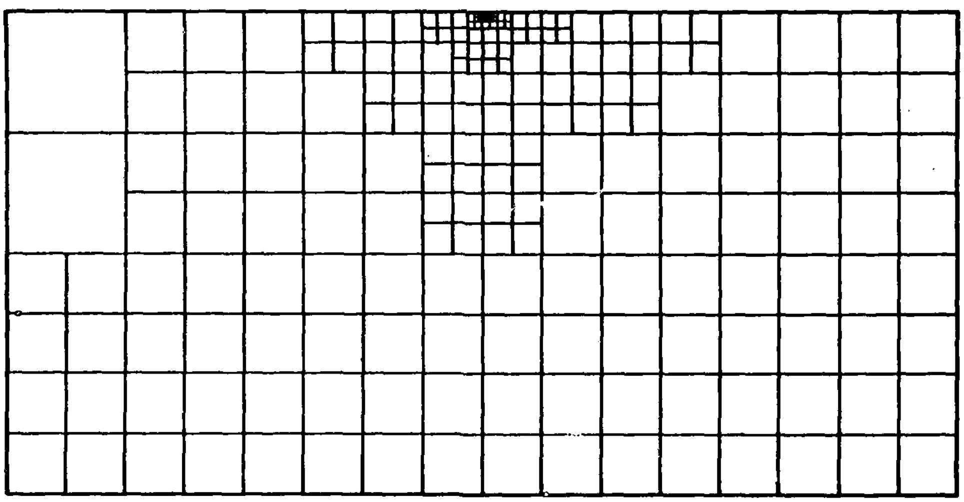


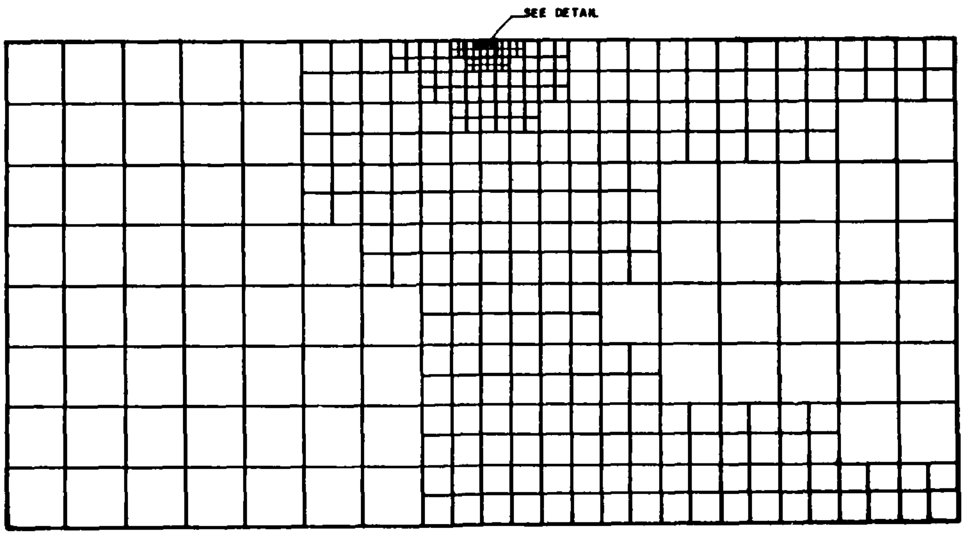












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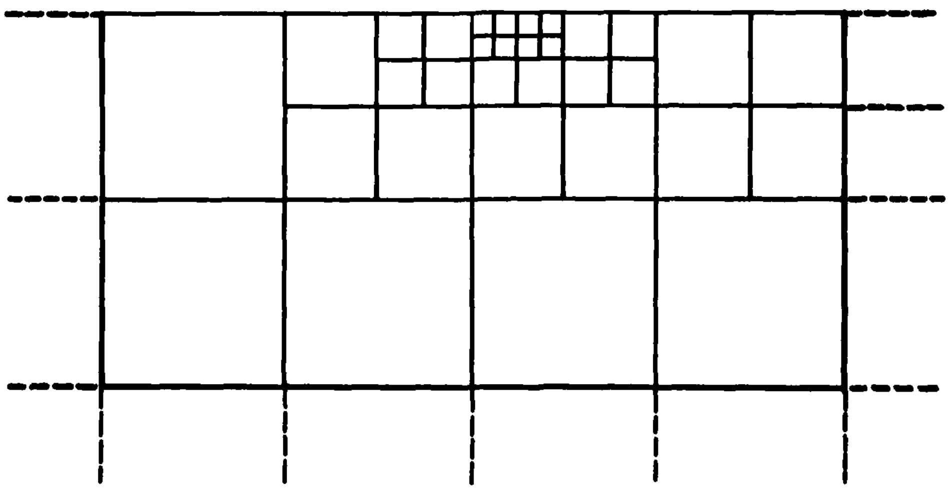

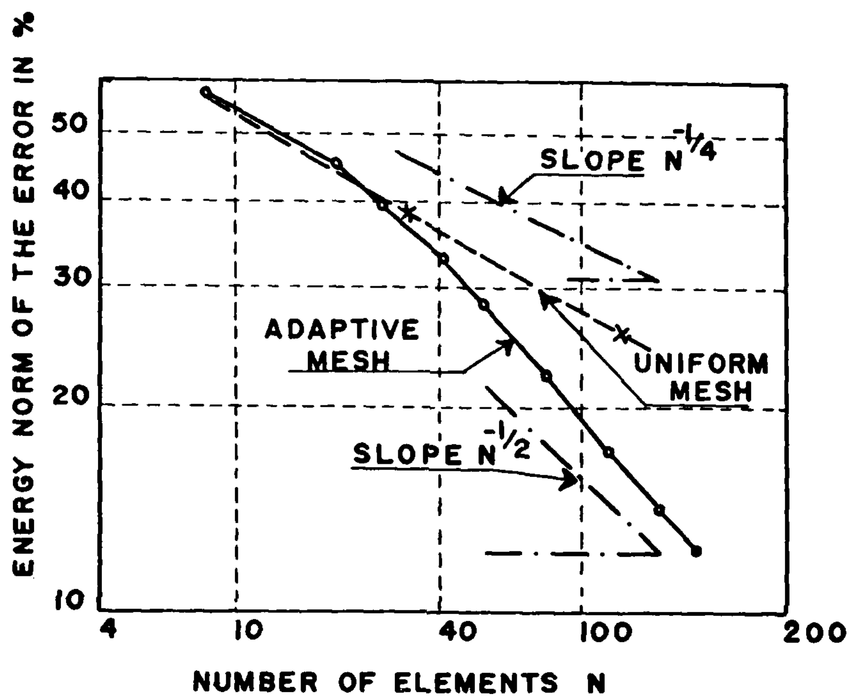

Figure 9.9. The energy norm of the error -- Example 2.

removes the influence of the singularities on the rate of convergence. We see also very clearly that using a uniform mesh we practically can never achieve an accuracy of 5%.

Figure 9.10 shows the behavior of the effectivity index for the Example 2. Once more we see that the effectivity index has practically acceptable value when the accuracy of the solution is in the range of 5-10%. In addition the rate of convergence of the effectivity index seems to be twice as high as that of the

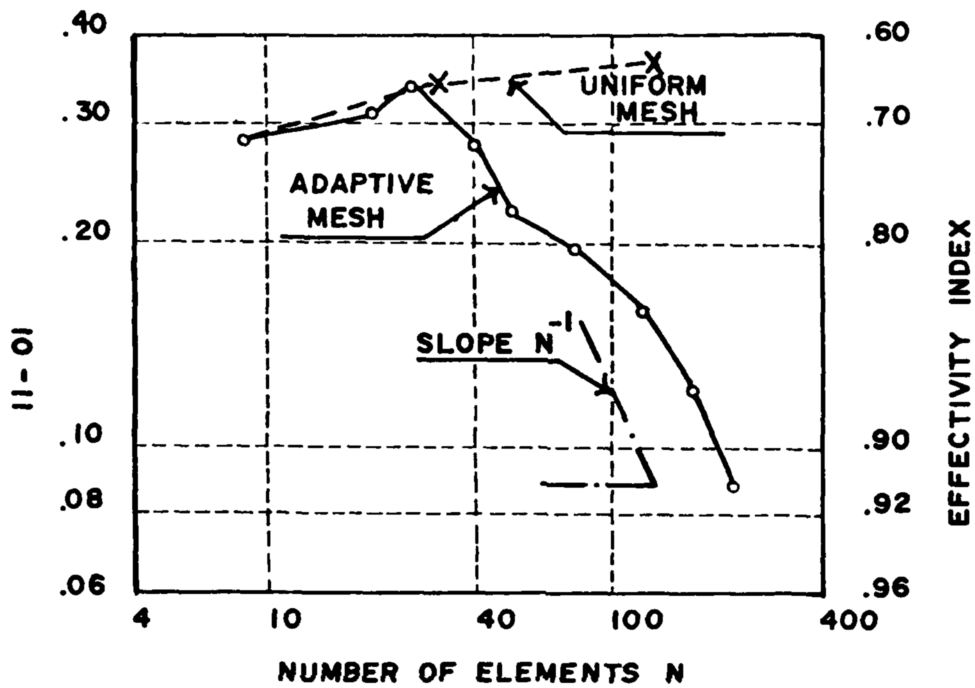

Figure 9.10. The effectivity index -- Example 2.

solution. Also we see that the error estimator performs much better for the adaptively constructed meshes which equilibrate the error indicators than for the uniform mesh.

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